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Modification Methods for Construction and Performance Analysis of Low-Density Parity-Check Codes over the Markov-Modulated Channels

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Abstract

We derive burst error correctable length of the low-density parity-check (LDPC) codes by the iterative decoding algorithms. By results of simulation, we show that the decoding performance of iterative decoding algorithms for the LDPC codes over the Markov-modulated binary symmetric channels (MM-BSC) are affected by the distance between elements (DBE). We also show that some column permuted LDPC codes that have larger DBE, have robustness in error performance over the MM-BSC, compared to the original LDPC codes.

1. Introduction

Iterative decoding algorithms of low-density parity-check (LDPC) codes [4] for channels with memory, such as the Markov-modulated channels (MMC), have been studied [3], [7] as well as for memoryless channels. These works have derived the iterative decoding algorithms combined with channel state estimation algorithm, such as the BCJR algorithm [1]. However, these works have not been much studied on the code structure of the LDPC codes.

In this paper, we consider the burst error correcting capability of the LDPC codes assuming the iterative decoding algorithms, such as the bit-flip (BF) and the sum-product (SP) decoding algorithms. We first introduce the distance between elements (DBE) [5] as a distance between elements of 1 at each rows of parity-check matrix of the LDPC codes. Then we derive burst error correctable length of the LDPC codes by the iterative decoding algorithms that depend on the DBEs. We also point out by results of simulation that some conventional construction methods with small value of the DBEs have weakness in error performance for the *Markov-modulated binary symmetric channels* (MM-BSC). We then propose a new modification method for construction of LDPC codes to make them robustness in error performance over the MM-BSC, compared to the original codes.

2. Preliminaries

2.1. LDPC codes [4]

Let $H = [H_{mn}]$ be a parity-check matrix whose row and column lengths are M and N, respectively, and $\boldsymbol{c} = (c_1, c_2, \ldots, c_N) \in \{0,1\}^N$ be a codeword of LDPC codes such that $\boldsymbol{c}H^{\mathrm{T}} = \boldsymbol{0}$. (N, w_r, w_c) LDPC codes have parity-check matrix with uniform weight w_r and w_c for each row and column, respectively. We define the following construction methods of LDPC codes.

Definition 1 (The Method $\mathcal{G}$). LDPC codes given by the *method* $\mathcal{G}$ consists of w_c pieces of $N/w_r (\triangleq \rho) \times N$ submatrices $H_G^{(i)}$, $i = [1, w_c]$, where row and column weight are w_r and 1, respectively [4]. The elements of first submatrix such that $H_G^{(1)} = [H_{G_{kn}}^{(1)}]$, $k \in [1, \rho]$, are given by

$$H_{G_{kn}}^{(1)} \triangleq \begin{cases} 1, & n \in [(k-1)w_r + 1, kw_r]; \\ 0, & \text{otherwise.} \end{cases} \qquad \square$$

Definition 2 (The Method $\mathcal{P}$). LDPC codes given by the *method* $\mathcal{P}$ consists of $w_r w_c$ pieces of $\rho \times \rho$ submatrices $H_P^{(i,j)}$, $j = [1, w_r]$, whose row and column weight are 1 [4]. □

A parity-check matrix is represented by the *bipartite graph*, which consists of two types of nodes called *check nodes* indexed by position of rows, and *symbol nodes* indexed by position of columns. A check node m and a symbol node n are connected with an edge if and only if $H_{mn} = 1$. A loop in the bipartite graph is a closed pass that starts from a symbol node and returns to the same symbol node through edges without passing the same edges more than once. A length of a loop is a number of edges of the closed pass.

2.2. The BF and the SP decoding algorithms

The BF and the SP decoding algorithms are iterative algorithms to estimate a transmitted codeword at symbol by symbol from a received sequence with updating two types of *messages* on the edge: messages from a symbol node to a check node and *vice versa*, respectively. We define the following sets for all (m,n), $m \in [1,M]$, $n \in [1,N]$, such that $H_{mn} = 1$.

$$\mathcal{A}(m) \triangleq \{n : H_{mn} = 1\} = \{n_{m,1}, n_{m,2}, \ldots, n_{m,w_r}\},$$
$$\mathcal{B}(n) \triangleq \{m : H_{mn} = 1\} = \{m_{n,1}, m_{n,2}, \ldots, m_{n,w_c}\},$$

where $n_{m,1} < n_{m,2} < \ldots < n_{m,w_r}$ and $m_{n,1} < m_{n,2} < \ldots < m_{n,w_c}$, respectively.

[The BF decoding algorithm]

In the case of the BF decoding algorithm, messages are defined by $r_{mn}^{(l)}$ and $v_{mn}^{(l)}$ where l denotes the number of iterations.

b1) For each (m,n) satisfying $H_{mn} = 1$, set $r_{mn}^{(0)} := y_n$. Set $l := 1$.

b2) For each (m,n) satisfying $H_{mn} = 1$, set $v_{mn}^{(l)}$ as following:

$$v_{mn}^{(l)} := \sum_{n' \in \mathcal{A}(m) \setminus n} r_{mn'}^{(l-1)}. \tag{1}$$

b3) For each (m,n) satisfying $H_{mn} = 1$, set $r_{mn}^{(l)}$ as following:

$$r_{mn}^{(l)} = \begin{cases} a, & \text{if } \forall m' \in \mathcal{B}(n) \setminus m,\ v_{m'n}^{(l)} = a; \\ y_n, & \text{otherwise.} \end{cases} \tag{2}$$

b4) For each n, set $\hat{c}_n^{(l)}$ as following:

$$\hat{c}_n^{(l)} = \begin{cases} a, & \text{if } \forall m \in \mathcal{B}(n),\ v_{mn}^{(l)} = a; \\ y_n, & \text{otherwise.} \end{cases} \tag{3}$$

b5) If $l = l_{max}$ or $\hat{\boldsymbol{c}} H^{\mathrm{T}} = \boldsymbol{0}$, then $\hat{\boldsymbol{c}}^{(l)}$ as an estimated sequence and stop the algorithm. Otherwise set $l := l + 1$ and go to b2).

[The SP decoding algorithm]

In the case of the SP decoding algorithm, messages are defined by $\alpha_{mn}^{(l)}$ and $\beta_{mn}^{(l)}$. The log likelihood ratio (LLR) is defined by $\lambda_n \triangleq \frac{\Pr(y_n|c_n=0)}{\Pr(y_n|c_n=1)}$.

s1) For each (m,n) satisfying $H_{mn} = 1$, set $\beta_{mn}^{(0)} := \lambda_n$. Set $l := 1$.

s2) For each (m,n) satisfying $H_{mn} = 1$, set $\alpha_{mn}^{(l)}$ as following:

$$\alpha_{mn}^{(l)} := 2\tanh^{-1}\Big(\prod_{n' \in \mathcal{A}(m) \setminus n} \tanh\big(2^{-1}\beta_{mn'}^{(l-1)}\big)\Big). \tag{4}$$

s3) For each (m,n) satisfying $H_{mn} = 1$, set $r_{mn}^{(l)}$ as following:

$$\beta_{mn}^{(l)} := \lambda_n + \sum_{m' \in \mathcal{B}(n) \setminus m} \alpha_{m'n}^{(l)}. \tag{5}$$

s4) For each n, set $\hat{c}_n^{(l)}$ as following:

$$\hat{c}_n^{(l)} = \begin{cases} 0, & \text{if } \lambda_n + \sum_{m \in \mathcal{B}(n)} \alpha_{mn}^{(l)} \geq 0; \\ 1, & \text{otherwise.} \end{cases} \tag{6}$$

s5) If $l = l_{max}$ or $\hat{\boldsymbol{c}} H^{\mathrm{T}} = \boldsymbol{0}$, then $\hat{\boldsymbol{c}}^{(l)}$ as an estimated sequence and stop the algorithm. Otherwise set $l := l + 1$ and go to s2).

2.3. MM-BSC

We assume a codeword of the LDPC code, denoted by $\boldsymbol{c}$ is transmitted through the MM-BSC. $\boldsymbol{c}$ is disturbed by additive noise $\boldsymbol{z} = (z_1, z_2, \ldots, z_N) \in \{0,1\}^N$ and the decoder receives a sequence $\boldsymbol{y} = \boldsymbol{c} \oplus \boldsymbol{z}$. The decoder estimates the transmitted codeword from the received sequence. The MM-BSC is hidden Markov model that consists of state set $\mathcal{S} = \{S_1, S_2, \ldots, S_{|\mathcal{S}|}\}$, with transition probability $p(s|s')$ from the state $s' \in \mathcal{S}$ to the state $s \in \mathcal{S}$, and the stational probability distribution $\boldsymbol{\pi} = (\pi_{s_1}, \pi_{s_2}, \ldots, \pi_{s_{|S|}})$. Following to the state transition of channels at time n, z_n is generated as channel noise. Let P_s be the error probability at state s, then the average error probability of the MM-BSC is $p_{ave} \triangleq \sum_{s \in \mathcal{S}} \pi_s P_s$.

3. Proposed modification method

3.1. Distance between elements

We define the *distance between elements* (DBE) as a distance between elements of 1 at each row of the parity-check matrix.

Definition 3 (The DBE). The DBEs $d_{m\gamma}$, $m \in [1,M]$, $\gamma \in [1, w_r - 1]$, and the minimum value of DBEs D_{min} are defined by the following equations, respectively:

$$d_{m\gamma} \triangleq n_{m,\gamma+1} - n_{m,\gamma}, \tag{7}$$
$$D_{min} \triangleq \min_{m,\gamma} d_{m\gamma}. \tag{8}$$

□

DBEs of the first submatrix of LDPC codes constructed by the method $\mathcal{G}$ are fixed to 1, so the distribution of DBEs is drawn to 1. The average value of DBEs of the LDPC codes constructed by the method $\mathcal{P}$ is maximum, so for a given parameter of (N, w_r, w_c), values of DBEs take relatively large [5]. Remark that the DBE is similar to the *minimum space distance* (MSD) [6] as a minimum value of zero-runs between elements of 1 for all rows of parity-check matrix.

3.2. Modification method for constructions of LDPC codes

For a parity-check matrix, the values of DBE are changed by column permutation. While column permuted parity-check matrices are equivalent to each other for memoryless channels in a sense that parameter of the code, such as length of the code, or weight distribution of the codewords are invariant. In addition, distribution of length of loops in the bipartite

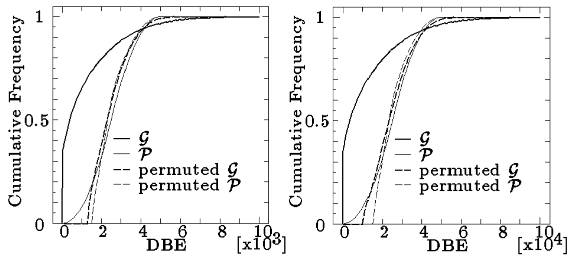

Figure 1: Cumulative frequency distribution of DBEs. The left and right figures are for N_2 and N_3, respectively.

graph is not changed, so the decoding performance over the memoryless channels of the BF and the SP decoding algorithms influenced by short loops, would also be unchanged.

Definition 4 (The Permuted $\mathcal{G}$ ($\mathcal{P}$)). For a given code constructed by the method $\mathcal{G}$ ($\mathcal{P}$), LDPC codes given by the *permuted* $\mathcal{G}$ ($\mathcal{P}$) is obtained by permuting columns of their parity-check matrix to have large minimum value of DBE. □

3.3. Cumulative frequency distribution of DBE

Figure 1 shows cumulative frequency distribution of DBEs of the $(N_i, 4, 3)$, $i = 2, 3$, LDPC codes constructed by the method $\mathcal{G}$ and $\mathcal{P}$, and the permuted $\mathcal{G}$ and $\mathcal{P}$, where $N_2 = 10^4$ and $N_3 = 10^5$, respectively. We can see from the figure that the cumulative frequency distribution of the method $\mathcal{G}$ differed from those of other methods. It is because that all of DBEs of the first submatrix of the method $\mathcal{G}$ are fixed to 1.

4. Some burst-error correcting capability at the first iteration

In this section, we will show some burst error correcting capability of the LDPC codes, assuming the BF and the SP decoding algorithms. In order to correct at the first iteration, an estimated sequence $\hat{\boldsymbol{c}}^{(1)}$ must be satisfied one of the following conditions:

S-A) $y_n \neq c_n$ and $\hat{c}_n^{(1)} = c_n$.

S-B) $y_n = c_n$ and $\hat{c}_n^{(1)} = c_n$.

4.1. The assumptions

We assume that a codeword $\boldsymbol{c}$ is disturbed by a burst error of length b. We also assume that the (N, w_r, w_c), $w_r > w_c > 1$, LDPC codes have no loops of length four, since at the first iteration of decoding procedures have not been affected by loops of length greater than four. When we consider burst error correcting capability of the LDPC codes by the BF and the SP decoding algorithms at the first iteration respectively, we should give some more detailed defini-

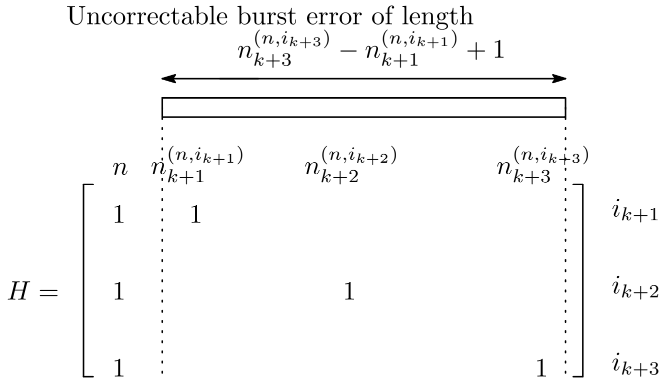

Figure 2: An example of uncorrectable burst error for y_n by the BF decoding algorithm when $w_c = 3$.

tion about the DBEs. Let $\mathcal{A}^{(1)}(n)$ be the index sets of column of the parity check matrix H defined as

$$\mathcal{A}^{(1)}(n) \triangleq \mathcal{A}(m') \setminus \{n\} = \{n_j^{(n,i_j)}\},$$
$$j = [1, \nu],\ m', i_j \in \mathcal{B}(n),$$

where $n_1^{(n,i_1)} < n_2^{(n,i_2)} < \cdots < n_\nu^{(n,i_\nu)}$ and $\nu \triangleq (w_r - 1)w_c$. The nth estimated symbol at first iteration $\hat{c}_n^{(1)}$ depends on received symbols $y_{n'}$, $n' \in \mathcal{A}^{(1)}(n)$.

Lemma 1. The BF and the SP decoding algorithms can correct y_n which is in error, after the first iteration when odd number of errors have occurred at $m' \in \mathcal{B}(n)$th parity-check equations. □

4.2. Burst-error correcting capability at the first iteration by the BF decoding algorithm

Lemma 2. The BF decoding algorithm cannot correct at the first iteration when even number of errors have occurred in at least one parity-check equations. □

From lemma 2, it is clear that the DBEs may have large value to correct long burst errors. From lemmas 1, 2, the condition S-A) is satisfied for $n \in [1, N]$ if and only if lemma 1 is satisfied.

Lemma 3. The *burst error correctable length of the BF decoding algorithm at the first iteration* (BECL-BF1) at symbol n denoted by $d_{min,n}^{(1)}$ satisfies the following equations:

$$d_{min,n}^{(1)} = \min_{\substack{k\in[0,\nu-w_c] \\ i_{k+1}\neq i_{k+2}\neq\cdots i_{k+w_c}}} \left\{ n_{k+w_c}^{(n,i_{k+w_c})} - n_{k+1}^{(n,i_{k+1})} \right\}. \tag{9}$$

And the BECL-BF1 denoted by $D_{min}^{(1)}$ satisfies the following equations:

$$D_{min}^{(1)} = \min\left\{ \min_{n\in[1,N]} \left\{ d_{min,n}^{(1)} \right\}, D_{min} \right\}. \tag{10}$$

Proof. Consider that w_c received symbols of positions $n_{k+1}^{(n,i_{k+1})}, n_{k+2}^{(n,i_{k+2})}, \ldots, n_{k+w_c}^{(n,i_{k+w_c})}$, $k = [0, \nu - w_c]$, such that $i_{k+1} \neq i_{k+2} \neq \cdots \neq i_{k+w_c}$ are in errors, then condition S-B) is not satisfied. So, burst error of length $n_{k+w_c}^{(n,i_{k+w_c})} - n_{k+1}^{(n,i_{k+1})} + 1$ which has occurred at positions $n_{k+1}^{(n,i_{k+1})}, n_{k+2}^{(n,i_{k+2})}, \ldots, n_{k+w_c}^{(n,i_{k+w_c})}$ cannot be corrected at the first iteration. So, equation (9) has been derived.

The equation (10) follows from the lemma 2. If a burst error of length $b > D_{min}$ has occurred, then at least one parity check equations may have even number of errors which cannot be corrected at the first iteration. From equation (9), BECL-BF1 corresponds to the smaller value of minimum value of $d_{min,n}^{(1)}$, $\forall n \in [1, N]$, and D_{min}. □

Figure 2 shows an example of uncorrectable burst error pattern when $w_c = 3$. Here, $i_{k+1}, i_{k+2}, i_{k+3}$ represent the positions of row and n, $n_{k+1}^{(n,i_{k+1})}$, $n_{k+2}^{(n,i_{k+2})}$, $n_{k+3}^{(n,i_{k+3})}$ represent the positions of column. A burst error of length $n_{k+3}^{(n,i_{k+3})} - n_{k+1}^{(n,i_{k+1})} + 1$ which has occurred at positions $n_{k+1}^{(n,i_{k+1})}, n_{k+1}^{(n,i_{k+1})} + 1, \ldots, n_{k+3}^{(n,i_{k+3})}$ cannot be corrected at the first iteration, since nth estimated symbol $\hat{c}_n^{(1)}$ would be $\hat{c}_n^{(1)} \neq y_n$.

Theorem 1. The (N, w_r, w_c) LDPC codes without loops of length four can correct a burst error of length $b \leq D_{min}^{(1)}$ by the BF decoding algorithm at the first iteration. □

4.3. Burst-error correcting capability at the first iteration by the SP decoding algorithm

Analysis of the SP decoding algorithm is more complicated than that of the BF decoding algorithm, since the SP decoding algorithm is a soft-input soft-output decoding algorithm. Lemma 2 is not always satisfied when w_c takes large value. In that case (*e.g.* $w_r = 5$, $w_c = 4$, and $p_{ave} \leq 0.0611861$), the SP decoding algorithm can correct errors at the first iteration when even number of errors have occurred in a parity-check equation. However, the LDPC codes with large w_c do not have better decoding performance in both the decoding error rate and its computational complexity, so we do not consider it to simplify the problem. As is the case of the BF decoding algorithm, from lemma 1 and the above mentioned argument, we only consider the condition that the y_n is not in error and the estimated sequence at position n at first iteration of the decoding algorithm is in error, again.

Lemma 4. The *burst error correctable length of the SP decoding algorithm at the first iteration* (BECL-SP1) at symbol n denoted by $s_{min,n}^{(1)}$ satisfies the following equations:

$$s_{min,n}^{(1)} = \min_{\substack{k \in [0, \nu - w_c + x] \\ i_{k+1} \neq i_{k+2} \neq \cdots i_{k+w_c-x}}} \left\{ n_{k+w_c-x}^{(n,i_{k+w_c-x})} - n_{k+1}^{(n,i_{k+1})} \right\}. \quad (11)$$

And the BECL-SP1 denoted by $S_{min}^{(1)}$ satisfies the following equations:

$$S_{min}^{(1)} = \min\left\{ \min_{n \in [1,N]} \left\{ s_{min,n}^{(1)} \right\}, D_{min} \right\}, \quad (12)$$

where x is a threshold value.

Proof. Consider that $w_c - x$ recieved symbols of positions $n_{k+1}^{(n,i_{k+1})}$, $n_{k+2}^{(n,i_{k+2})}, \ldots, n_{k+w_c-x}^{(n,i_{k+w_c-x})}$, $k \in [0, \nu - w_c + x]$, such that $i_{k+1} \neq i_{k+2} \neq \cdots \neq i_{k+w_c-x}$ are in errors, then condition S-B) is satisfied, where x is a threshold value (it is explained in appendix in detail). So, burst error of length $n_{k+w_c-x}^{(n,i_{k+w_c-x})} - n_{k+1}^{(n,i_{k+1})} + 1$ which has occurred at positions $n_{k+1}^{(n,i_{k+1})}$, $n_{k+2}^{(n,i_{k+2})}, \ldots$, $n_{k+w_c-x}^{(n,i_{k+w_c-x})}$ cannot be corrected at the first iteration. Therefore, equation (11) holds.

The equation (12) follows from the lemma 2. If burst error of length $b > S_{min}$ has occurred, then at least one parity check equations may have even number of errors which cannot be corrected at the first iteration. From equation (11), BECL-SP1 corresponds to the smaller value of minimum value of $s_{min,n}^{(1)}$, $\forall n \in [1, N]$, and D_{min}. □

Theorem 2. The (N, w_r, w_c) LDPC codes without loops of length four can correct a burst error of length $b \leq B_{min}^{(1)}$ by the SP decoding algorithm at the first iteration. □

5. Results of simulation and discussion

We will show by simulations that LDPC codes with small minimum value of DBEs are inferior to those with large minimum value of DBEs in error performance over the MM-BSC.

5.1. Conditions

5.1.1. Construction method

We assume (N_i, w_r, w_c), $i = 1, 2, 3$, LDPC codes with $N_1 = 10^3$, $N_2 = 10^4$, $N_3 = 10^5$. For each i, we compared (N_i, w_r, w_c) LDPC codes constructed by the method $\mathcal{G}$ and $\mathcal{P}$ with that by the permuted $\mathcal{G}$ and $\mathcal{P}$. We generated three parity-check matrices by the method $\mathcal{G}$ and $\mathcal{P}$. Moreover, we constructed column permuted parity-check matrices (by the proposed modification method) from above constructed LDPC codes to have large minimum value of DBE.

5.1.2. Decoding algorithms

We used the BF and the SP decoding algorithms. In the case of BF decoding algorithm, we used $(N_i, 4, 3)$ LDPC codes with $i = 2, 3$, and in the case of SP decoding algorithm, $(N_i, 4, 3)$ LDPC codes with $i = 1, 2$. The

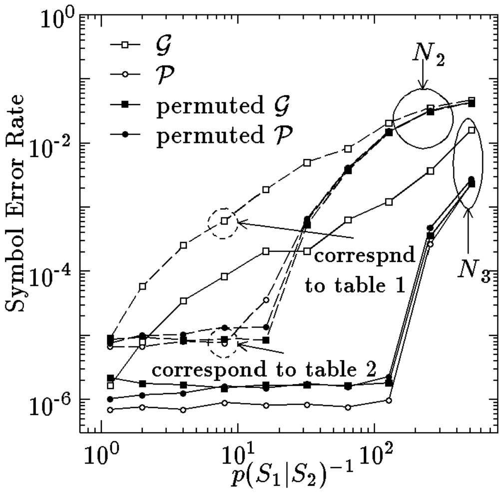

Figure 3: Simulation results by the BF decoding algorithm at p_{ave}=0.07. Points showed "table 1" and "table 2" in Fig. 3 correspond to those shown in table 1 and table 2, respectively.

Table 1: The MER and SER of the method $\mathcal{G}$.

l	First	Second	Third	SER
1	5.33E-02	4.96E-02	4.96E-02	4.65E-02
2	3.45E-02	4.38E-02	4.39E-02	3.64E-02
3	2.96E-02	3.33E-02	3.33E-02	3.11E-02
98	3.31E-04	5.11E-04	5.15E-04	6.15E-04
99	3.32E-04	5.14E-04	5.04E-04	6.13E-04
100	3.26E-04	5.09E-04	5.05E-04	6.08E-04

simulations of the BF (SP) decoding algorithm continued until 2×10^4 (10^5) codewords were transmitted or numbers of uncorrected codewords reached 50. We used the average of error probability of the MM-BSC for the SP decoding algorithm. The maximum number of iterations was fixed to 100 for both decoding algorithms.

5.1.3. MM-BSC

We assumed the MM-BSC with $|\mathcal{S}| = 2$. Average of error probability of the MM-BSC is $p_{ave} = 0.07$ for the BF decoding algorithm and $p_{ave} = 0.15$ for the SP decoding algorithm. We set the error probability at each state as $P_{S_1} = 0$, $P_{S_2} = 0.5$, and changed transition probabilities $p(S_2|S_1)$, $p(S_1|S_2)$ with keeping p_{ave} constant.

5.2. Results of simulation

Figure 3 and 4 show results of simulation by the BF and the SP decoding algorithm. The horizontal axis represents $p(S_1|S_2)^{-1}$, the average of recurrence time at state S_2, and the vertical axis represents *symbol error rate* (SER). The leftmost plotted points in both

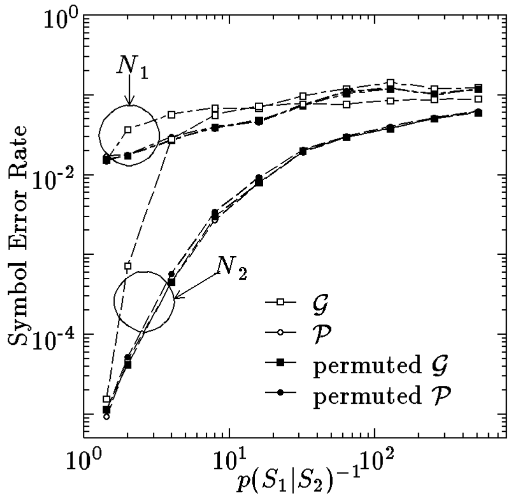

Figure 4: Simulation results by the SP decoding algorithm at p_{ave}=0.15.

Table 2: The MER and SER of the permuted $\mathcal{G}$.

l	First	Second	Third	SER
1	5.40E-02	5.40E-02	5.40E-02	3.73E-02
2	3.87E-02	3.87E-02	3.88E-02	2.93E-02
3	2.55E-02	2.55E-02	2.55E-02	2.16E-02
98	7.45E-06	7.50E-06	7.21E-06	8.62E-06
99	7.27E-06	7.60E-06	7.29E-06	8.81E-06
100	7.48E-06	7.54E-06	7.43E-06	8.64E-06

figures satisfy $p(S_1|S_2) + p(S_2|S_1) = 1$ where the MM-BSC does not have memory, and hence, is equivalent to the BSC. So decoding performance of the LDPC codes constructed by four methods were all identical. For the plotted points except for the leftmost points, the MM-BSC has memory. If $p(S_1|S_2)^{-1}$ becomes larger, it has longer memory.

Table 1 and 2 show the *message error rate* (MER) as error rate of messages $r_{mn}^{(l)}$ of the first, the second, and the third submatrices of $(N_2, 4, 3)$ LDPC codes constructed by the method $\mathcal{G}$ and the permuted $\mathcal{G}$ when $p(S_1|S_2)^{-1} = 2^3$ by the BF decoding algorithm, respectively. The first column of each tables shows the number of iteration denoted by l. The second, the third, and the fourth columns show the MERs per iterations of each submatrices and the fifth column shows SERs per iterations.

5.3. Discussions

We see that performance of the LDPC codes constructed by the method $\mathcal{G}$ are inferior to codes constructed by the other methods, including the permuted $\mathcal{G}$. It can be concluded from figure 1 that only the method $\mathcal{G}$ differed from the other methods. In figure 3, the SER of all the methods except the method $\mathcal{G}$ at $2^1 \leq p(S_1|S_2)^{-1} \leq 2^4$ when $N = N_2$ and at

$2^1 \le p(S_1|S_2)^{-1} \le 2^7$ when $N = N_3$, were almost same and constant. This means that the MM-BSC at this range can be regarded as BSC.

In section 4, we have derived burst error correcting capability at the first iteration for both decoding algorithms which depend on the DBEs. Recall that the DBEs of a first submatrix of the method $\mathcal{G}$, $H_G^{(1)}$ are all 1, so the even number of errors have been likely occurred in a parity check equation. From lemma 2, it cannot be corrected at the first iteration by the BF decoding algorithm. From table 1, the MER of the first submatrix of the method $\mathcal{G}$ differed from those of the other submatrices. However, from table 2, the MERs of all the submatrices of the permuted $\mathcal{G}$ were almost identical to each other. And the SERs per iteration of the method $\mathcal{G}$ were inferior to those of the permuted $\mathcal{G}$.

The threshold value takes $x = 0$ when $w_c = 3$ and $p_{ave} = 0.07$, so the SP decoding algorithm cannot also correct at the first iteration. We have confirmed that the SERs per numbers of the iteration of the method $\mathcal{G}$ at figure 3 and 4 were also inferior to those of the other methods.

6. Conclusion and remarks

For the MM-BSC, we have proposed a new modification method of LDPC codes to have large values of DBE. We have derived the burst error correctable length of the LPDC codes by the BF and the SP decoding algorithms at the first iteration, respectively. Then we show by some results of simulation that proposed method have robustness in error performance over the MM-BSC, assuming iterative decoding algorithms.

As further works, the derivations of the burst error correctable length of iterations greater than one are needed. We also need to derive the burst error correctable length when two or more burst errors are occurred in one codeword. Moreover, analysis for the code with stopping sets [2] should be considered.

Acknowledgments

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Appendix : Derivation of a threshold value x

Consider that a transmitted symbol $c = 0$ and its received symbol $y = 0$ (the corresponding LLR takes $\lambda \triangleq \frac{\Pr(y|c)}{\Pr(y|c\oplus 1)} > 0$), then its estimated symbol at first iteration of the SP decoding algorithm denoted by $\hat{c}^{(1)}$, is given by the following equation:

$$\hat{c}^{(1)} = \begin{cases} 0, & \text{if } \lambda + (w_c - e)\alpha - e\alpha \ge 0; \\ 1, & \text{otherwise,} \end{cases} \quad (13)$$

where α be the log extrinsic value ratio (LER) which is denoted by

$$\alpha \triangleq 2\tanh^{-1}\left(\left(\tanh\left(2^{-1}\lambda\right)\right)^{w_r - 1}\right),$$

and e is the number of LER taking value $-\alpha$. So, we must derive the threshold value x which is given by

$$\lambda + (w_c - x)\alpha - x\alpha \ge 0, \qquad w_c \ge x,\ x \ge 0, \quad (14)$$

The equation (14) indicates that $w_c - x$ check nodes have sent messages (which are correspond to α) whose sign is the same as that of λ. It also indicates that x check nodes have sent messages (which are correspond to $-\alpha$) whose sign is different from that of λ. After manuscripting the equation (14), we get

$$x = \begin{cases} 0, & \text{if } \left\lfloor \frac{\lambda + w_c\alpha}{2\alpha} \right\rfloor \le 0; \\ \left\lfloor \frac{\lambda + w_c\alpha}{2\alpha} \right\rfloor, & \text{otherwise.} \end{cases} \quad (15)$$

The same argument is also valid when $c = 1$, $y = 1$, if $\Pr(c = 0) = \Pr(c = 1) = 0.5$ is assumed. □

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