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A Note on the Construction of Nonlinear Unequal Orthogonal Arrays from Error-Correcting Codes

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Abstract Orthogonal arrays have been used in the field of experimental design. Hedayat and Sloane showed the relation between orthogonal arrays and error-correcting codes[1]. And they proposed some construction methods of both linear and nonlinear orthogonal arrays from error-correcting codes. On the other hand, the paper[5] defined unequal orthogonal arrays as new class. It showed that unequal orthogonal arrays are more applicable to experimental design. Furthermore, it showed the relation between unequal orthogonal arrays and unequal error-correcting codes[3], and proposed the construction method of unequal orthogonal arrays from unequal error-correcting codes. But orthogonal arrays from this construction method are all linear. In this paper, we clarify the relation between nonlinear unequal orthogonal arrays and codes. And we propose one of construction methods of nonlinear unequal orthogonal arrays from error-correcting codes.

Key words Experimental Design, Orthogonal Arrays, Unequal Error-correcting Codes, Nonlinear Codes

1 Introduction

Experimental design has been used in many fields, for example in quality management. In experimental design, it is important to design experiments so that we can estimate the effects of factors and their interactions where the number of experiments is as few as possible.

In order to reduce the number of experiments, constructing orthogonal arrays is important[1]. Generally, there are two classes of orthogonal arrays: one is linear orthogonal arrays, the other is nonlinear orthogonal arrays.

Hedayat and Sloane showed the relation between orthogonal arrays and error-correcting codes. And they proposed some construction methods of orthogonal arrays from error-correcting codes[1]. According to their result, its relation can be divided into two types: the first is the relation between linear orthogonal arrays and linear codes, the second is the one between nonlinear orthogonal arrays and nonlinear codes. Furthermore, they proposed some construction methods of orthogonal arrays from linear and nonlinear error-correcting codes.

On the other hand, the paper[5] defined *unequal orthogonal arrays* as new class. From the point of view of the definition, the class with which Hedayat and Slone dealt is *equal orthogonal arrays*. It showed that unequal orthogonal arrays could reduce the number of experiments when we have knowledge about effects of interactions in detail. Furthermore, it showed the relation between unequal arrays and unequal error-correcting codes[3], and proposed construction method of unequal orthogonal arrays from unequal error-correcting codes. But orthogonal arrays from this construction method are all linear.

Table 1: $OA(4,3,2,2)$

0	0	0
1	1	0
0	1	1
1	0	1

In this paper, we clarify the relation between nonlinear unequal orthogonal arrays and codes. Furthermore, we extend the construction method of linear unequal orthogonal arrays, and propose one of construction methods of nonlinear unequal orthogonal arrays.

2 Orthogonal Arrays

2.1 Orthogonal Arrays

Definition2.1:[1] An $M \times n$ array A with entries from $GF(s)$ is said to be an *orthogonal array with s levels and strength t* if every $M \times t$ subarray of A contains each t-tuple based on $GF(s)$ exactly same times as row. We will denote such an array by $OA(M,n,s,t)$.

Example2.1: The array in Table 1 is orthogonal array with strength 2. It is an $OA(4,3,2,2)$.

We consider only the case of $s=2$.

An orthogonal array $OA(M,n,2,t)$ is said to be linear if the rows of $OA(M,n,2,t)$ form a linear vector space. If an orthogonal array $OA(M, n, 2, t)$ is linear, $OA(M, n, 2, t)$ has a basis for the linear vector space. This basis is given in the form of $(\log_2 M) \times n$ matrix called generator matrix.

Next, we show the case that we can reduce the number of experiments when an orthogonal array is applied to an experimental design. Now we consider the case that there is a response variable of interest. And there are three factors F_1, F_2 and F_3 that might affect the response variable. Where each F_i has level $F_{i,0}$ and level $F_{i,1} (i = 1, 2, 3)$. Then we examine how changes in the levels of the factors affect the response variable. For example, this case is the following case. There is a ratio of defective products as a response variable. And there are three factors that might affect the response variable; the type of catalyst, machine, and material. Where each factor has two levels; catalyst0 and catalyst1, machine0 and machine1 ,and material0 and material1. Then we examine how changes in the levels of the factors affect the response variable.

In the above case, we can estimate all the main effects and all the interaction effects of two factors with the following experimental design:

$$\begin{array}{c} (F_{1,0}F_{2,0}F_{3,0}) \\ (F_{1,0}F_{2,0}F_{3,1}) \\ (F_{1,0}F_{2,1}F_{3,0}) \\ (F_{1,0}F_{2,1}F_{3,1}) \\ (F_{1,1}F_{2,0}F_{3,0}) \\ (F_{1,1}F_{2,0}F_{3,1}) \\ (F_{1,1}F_{2,1}F_{3,0}) \\ (F_{1,1}F_{2,1}F_{3,1}) \end{array} \tag{1}$$

Where the main effect of the factor F_i is measurements which represent how change in the levels of the factor F_i affects the response variable, and the interaction effect of the factors F_i and F_j is measurements which represent how changes in the level combinations of the factors F_i and F_j affect the response variable. Generally, an interaction effect of k factors is measurements which represent how changes in the level combinations of the k factors affect a response variable. And each vector of Eq.(1) corresponds to one time of experiments. For example, $(F_{1,0}, F_{2,0}, F_{3,0})$ correspond to the experiment whose level combination is $F_{1,0}, F_{2,0}, F_{3,0}$. This experimental design contains all level combinations as the vectors. The design such as Eq.(1) is said to be complete design.

Further, we suppose that we know that there is no interaction effect of two factors by experiences. Then we can estimate all the main effect of factors with the following experimental design:

$$\begin{array}{c} (F_{1,0}F_{2,0}F_{3,0}) \\ (F_{1,0}F_{2,1}F_{3,1}) \\ (F_{1,1}F_{2,0}F_{3,1}) \\ (F_{1,1}F_{2,1}F_{3,0}) \end{array} \tag{2}$$

This experimental design is made using the orthogonal array $OA(4, 3, 2, 2)$ we show in Table 1. Where, each row of the $OA(4, 3, 2, 2)$ corresponds to the vectors in Eq.(2). For example, 000 which is the first row of the $OA(4, 3, 2, 2)$ correspond to $(F_{1,0}F_{2,0}F_{3,0})$. Then we can reduce the number of experiments using the design that is Eq.(2).

As the above case, when we have knowledge of interaction effects, we can reduce the number of experiments using an orthogonal array. Generally, when an orthogonal array $OA(M, n, 2, t)$ is applied to experimental design, the number of experiments is M, and we can estimate interaction effects of at most $\lfloor \frac{t}{2} \rfloor$ factors.

Next we describe a necessary and sufficient condition for an array to be an orthogonal array.

Theorem 2.1:[1] An $M \times n$ array A with 0, 1 entries is an $OA(M, n, 2, t)$ if and only if

$$\sum_{u=row of A} (-1)^{u \cdot v} = 0,$$

for all 0, 1 vectors v containing w 1's, for all w in the range $1 \leq w \leq t$, where the sum is over all rows u of A.

2.2 Orthogonal Arrays From Codes

2.2.1 Linear Orthogonal Arrays from Linear Codes

Let $w(u)$ be the Hamming weight of a vector $u = (u_1, u_2, \ldots, u_n)$. An error-correcting code or simply code is any collection C of vectors in $GF(s)^n$. The vectors in C are called codewords. Let $dist(u, v)$ be the Hamming distance between two vectors u,v. We define the minimal distance d of a code C to be the minimal distance between distinct codewords:

$$d = \min_{u,v \in C, u \neq v} dist(u, v).$$

If C contains M codewords, then we say that it is a code of length n, size M and minimal distance d over $GF(s)$ or simply $(n, M, d)_s$code. We consider the case of $s = 2$ as orthogonal arrays.

C is said to be linear if C is a linear vector subspace. If C is linear, C has the dual code $C^{\perp}$. Let $d^{\perp}$ be the minimal distance of $C^{\perp}$. Then $d^{\perp}$ is said to be a dual distance of C.

Theorem2.2:[1] If C is a $(n, M, d)_2$ linear code over $\{0, 1\}$ with dual distance $d^{\perp}$ then the codewords of C form the rows of an $OA(M, n, 2, d^{\perp} - 1)$ with entries from $\{0, 1\}$.

2.2.2 Nonlinear Orthogonal Arrays from Nonlinear Codes

The Group Algebra

we are going to describe binary vectors of length n by polynomials in $z_1, z_2, \ldots, z_n$. For example, $100\ldots0$ will be represented by z_1, $1010\ldots0$ by z_1z_3 and so on. In general $\boldsymbol{v} = v_1v_2\ldots v_n$ is represented by $z_1^{v_1}z_2^{v_2}\ldots z_n^{v_n}$, which we abbreviate $\boldsymbol{z}^{\boldsymbol{v}}$. We make the convention that $z_i^2 = 1$ for all i. This makes the set of all $\boldsymbol{z}^{\boldsymbol{v}}$ into a multiplicative group denoted by G. Thus $\{0,1\}^n$ and G are isomorphic groups. with addition in $\{0,1\}^n$

$$\begin{aligned} \boldsymbol{v}+\boldsymbol{w} &= (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) \\ &= (u_1+v_1, u_2+v_2, \ldots, u_n+v_n), \end{aligned}$$

corresponding to multiplication in G:

$$\boldsymbol{z}^{\boldsymbol{v}}\boldsymbol{z}^{\boldsymbol{w}} = z_1^{v_1}z_2^{v_2}\cdots z_n^{v_n}\cdot z_1^{w_1}z_2^{w_2}\cdots z_n^{w_n} = \boldsymbol{z}^{\boldsymbol{v}+\boldsymbol{w}}.$$

Definition2.2:[2] The *group algebra* QG of G over the rational numbers Q consists of all formal sums

$$\sum_{\boldsymbol{v}\in\{0,1\}^n} a_{\boldsymbol{v}}\boldsymbol{z}^{\boldsymbol{v}}, a_{\boldsymbol{v}} \in Q, \boldsymbol{z}^{\boldsymbol{v}} \in G.$$

Addition and multiplication of elements of QG are defined in the natural way by

$$\sum_{\boldsymbol{v}\in\{0,1\}^n} a_{\boldsymbol{v}}\boldsymbol{z}^{\boldsymbol{v}} + \sum_{\boldsymbol{v}\in\{0,1\}^n} b_{\boldsymbol{v}}\boldsymbol{z}^{\boldsymbol{v}} = \sum_{\boldsymbol{v}\in\{0,1\}^n} (a_{\boldsymbol{v}}+b_{\boldsymbol{v}})\boldsymbol{z}^{\boldsymbol{v}},$$

$$r\sum_{\boldsymbol{v}\in\{0,1\}^n} a_{\boldsymbol{v}}\boldsymbol{z}^{\boldsymbol{v}} = \sum_{\boldsymbol{v}\in\{0,1\}^n} ra_{\boldsymbol{v}}\boldsymbol{z}^{\boldsymbol{v}}, r \in Q,$$

and

$$\sum_{\boldsymbol{v}\in\{0,1\}^n} a_{\boldsymbol{v}}\boldsymbol{z}^{\boldsymbol{v}} \cdot \sum_{\boldsymbol{v}\in\{0,1\}^n} b_{\boldsymbol{w}}\boldsymbol{z}^{\boldsymbol{v}} = \sum_{\boldsymbol{v},\boldsymbol{w}\in\{0,1\}^n} a_{\boldsymbol{v}}b_{\boldsymbol{w}}\boldsymbol{z}^{\boldsymbol{v}}.$$

To each $\boldsymbol{u} \in \{0,1\}$, we associate the mapping $\chi_{\boldsymbol{u}}$ from G to the rational numbers given by

$$\chi_{\boldsymbol{u}}(\boldsymbol{z}^{\boldsymbol{v}}) = (-1)^{\boldsymbol{u}\cdot\boldsymbol{v}},$$

where $\boldsymbol{u}\cdot\boldsymbol{v}$ is the scalar product of the vectors $\boldsymbol{u}, \boldsymbol{v}$ over Q. $\chi_{\boldsymbol{u}}$ is called a character of G. $\chi_{\boldsymbol{u}}$ is extended to act on QG by linearity.

$$\begin{aligned} \chi_{\boldsymbol{u}}\Big(\sum_{\boldsymbol{v}\in\{0,1\}^n} a_{\boldsymbol{v}}\boldsymbol{z}^{\boldsymbol{v}}\Big) &= \sum_{\boldsymbol{v}\in\{0,1\}^n} a_{\boldsymbol{v}}\chi_{\boldsymbol{u}}(\boldsymbol{z}^{\boldsymbol{v}}) \\ &= \sum_{\boldsymbol{v}\in\{0,1\}^n} (-1)^{\boldsymbol{u}\cdot\boldsymbol{v}}a_{\boldsymbol{v}}. \end{aligned}$$

Let

$$\gamma = \sum_{\boldsymbol{v}\in\{0,1\}^n} c_{\boldsymbol{v}}\boldsymbol{z}^{\boldsymbol{v}},$$

be an arbitrary element of the group algebra QG with the property that

$$M = \sum_{\boldsymbol{v}\in\{0,1\}^n} c_{\boldsymbol{v}} \neq 0.$$

We call the $(n+1)$-tuple $\{A_0, A_1, \ldots, A_n\}$, where

$$A_i = \sum_{w(\boldsymbol{v})=i} c_{\boldsymbol{v}},$$

the weight distribution of γ.

Definition2.3:[2] The *transform* of γ is the element $\gamma^{\perp}$ of QG given by

$$\gamma^{\perp} = \frac{1}{M}\sum_{\boldsymbol{u}\in\{0,1\}^n} \chi_{\boldsymbol{u}}(\gamma)\boldsymbol{z}^{\boldsymbol{u}}.$$

Nonlinear Orthogonal Arrays from Codes

Now let C be a linear or nonlinear $(n, M, d)_2$code. C is described by the element

$$\gamma = \sum_{\boldsymbol{v}\in C} \boldsymbol{z}^{\boldsymbol{v}},$$

of QG. Let $\delta = \frac{1}{M}\gamma^2$, and $\delta^{\perp}$ is the transform of δ. The weight distribution of $\delta^{\perp}$ is $\{B_0^{\perp}, B_1^{\perp}, \ldots, B_n^{\perp}\}$, where

$$B_i^{\perp} = \frac{1}{M}\sum_{w(\boldsymbol{u})=i} \chi_{\boldsymbol{u}}(\delta).$$

Definition2.4:[2] The *dual distance* $d^{\perp}$ of a code C is defined by $B_i^{\perp} = 0 for 1 \leq i \leq d^{\perp}-1$, $B_{d^{\perp}}^{\perp} \neq 0$, If C is linear, $d^{\perp}$ is the minimum distance of $C^{\perp}$.

Theorem2.3:[1][2] If C is a $(n, M, d)_2$ code over $\{0,1\}$ with dual distance $d^{\perp}$ then the codewords of C form the rows of an $OA(M, n, 2, d^{\perp}-1)$ with entries from $\{0,1\}$.

3 Unequal Orthogonal Arrays

3.1 Unequal Orthogonal Arrays

Definition3.1: An $M \times n$ array A with $0, 1$ entries is said to be an *unequal orthogonal array with* 2 *levels and strength* $r = (r_1, r_2, \ldots, r_n)$ if every $M \times r_i$ subarray of A, which contain ith column of A, contains each r_i-tuple based on $\{0,1\}$ exactly same times as row. We will denote such an array by $OA(M, n, 2, r)$.

When $OA(M, n, 2, (r_1, r_2, \ldots r_n))$ is applied to experimental design, we can estimate the effects of interactions of at most$[\frac{r_i}{2}]$ factors which contains ith factor. It was shown that there are cases that unequal array reduce more numbers of experiments than equal orthogonal arrays[5].

For example, this case is the following case. Here, Let $F_i \times F_j$ be the interaction of F_i and F_j. We suppose that there are three factors F_1, F_2 and F_3. And we know that there are $F_1 \times F_2$ and $F_1 \times F_3$. When an equal orthogonal array $OA(M_1, 3, 2, 4)$ is used, we can estimate not only $F_1 \times F_2, F_1 \times F_3$ but $F_2 \times F_3$, although we need not estimate $F_2 \times F_3$. On the other hand, when an unequal orthogonal array $OA(M_2, 3, 2, (4, 2, 2))$ is used, we can not estimate $F_2 \times F_3$. Therefore unequal orthogonal array reduce the number of experiments.

Next we describe a necessary and sufficient condition for an array to be an unequal orthogonal array.

Theorem 3.1: An $M \times n$ array A with 0, 1 entries is an $OA(M, n, 2, (r_1, r_2, \ldots, r_n))$ if and only if

$$\sum_{u = row of A} (-1)^{u \cdot v} = 0,$$

for all 0, 1 vectors $v = (v_1, v_2, \ldots, v_n)$ such that $v_i \neq 0$ and $w(v) = w$ for all w in the range $1 \leq w \leq r_i$, where the sum is over all rows u of A.

3.2 Unequal Orthogonal Arrays from Code

3.2.1 Linear Unequal Orthogonal Arrays from Codes

Linear Unequal Orthogonal Arrays from Codes

The separation $(d_1, d_2, \ldots, d_n)$ of linear code C is defined by

$$\begin{aligned} d_i = \min\{dist(u, v) \mid u &= (u_1, u_2, \ldots, u_n), \\ v &= (v_1, v_2, \ldots, v_n), u, v \in C, u_i \neq v_i\}, \\ &\quad for \quad i = 1, 2, \ldots, n. \end{aligned}$$

Let $(d_1^{\perp}, d_2^{\perp}, \ldots, d_n^{\perp})$ be the separation of $C^{\perp}$ which is the dual code of C. Then $(d_1^{\perp}, d_2^{\perp}, \ldots, d_n^{\perp})$ is said to be a dual separation of C.

Theorem3.2: If C is a $(n, M, d)_2$linear code over $\{0, 1\}$ with dual separation $(d_1^{\perp}, d_2^{\perp}, \ldots, d_n^{\perp})$, then the codewords of C form the row of an $OA(M, n, 2, (d_1^{\perp} - 1, d_2^{\perp} - 1, \ldots, d_n^{\perp} - 1))$ with entries from {0,1}.

The Construction Method

Now we show the construction method of orthogonal array. This construction method is the method that is applied the construction method of unequal error-correcting code[3] to.

The Construction Method1: Let there be two generator matrix of orthogonal arrays; G_1 is the generator matrix of a linear orthogonal array $OA(M_1, n_1, 2, t_1)$, and G_2 is the one of a linear orthogonal array$OA(M_2, n_2, 2, t_2)$, where $t_2 \leq t_1$. Let G_1 and G_2 be joined as submatrices of G where G_1 and G_2 overlap, as shown in Figure 1. The orthogonal array for which G is generator matrix is $(M_1 M_2) \times (n_1 + n_2 - n_{0L})$ array. Let $n_{0L} \leq t_2/2$.

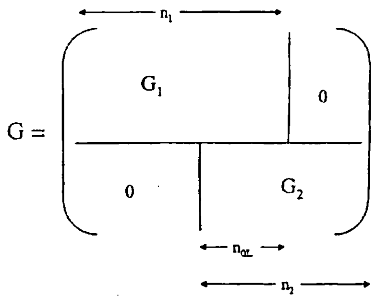

Figure 1: The construction method of linear orthogonal array

Theorem3.3: The orthogonal arrays for which G is generator matrix is $OA(M_1 M_2, n_1 + n_2 - n_{0L}, 2, (r_1, r_2, \ldots r_n))$, where

$$\begin{aligned} r_i &\geq t_1 \quad (i = 1, 2, \ldots, n_1 - n_{0L}), \\ r_i &\geq t_2 \quad (i = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 + n_{0L}), \\ r_i &\geq t_1 + t_2 - n_{0L} \quad (i = n_1 - n_{0L} + 1, \ldots n_1). \end{aligned}$$

4 Nonlinear Unequal Orthogonal Arrays

Nonlinear Unequal Orthogonal Arrays from Codes

Let C be a linear or nonlinear $(n, M, d)_2$code and

$$\gamma = \sum_{v \in C} z^{v}.$$

Let $\delta = \frac{1}{M}\gamma^2$ and $\delta^{\perp}$ is the transform of δ. Now $(B_{i,1}^{\perp}, B_{i,2}^{\perp}, \ldots, B_{i,n}^{\perp})$, $for\ i = 1, 2, \ldots, n$ is defined by

$$B_{i,j}^{\perp} = \frac{1}{M} \sum_{u_i \neq 0, w(u) = j} \chi u(\delta).$$

Definition4.1: The *dual separation* $(d_1^{\perp}, d_2^{\perp}, \ldots, d_n^{\perp})$ of a code C is defined by

$$\begin{aligned} &B_{i,j}^{\perp} = 0, \ for\ 1 \leq j \leq d_i^{\perp} - 1, \\ &B_{d^{\perp}}^{\perp} \neq 0. \end{aligned}$$

If C is linear, $(d_1^{\perp}, d_2^{\perp}, \ldots, d_n^{\perp})$ is the separation of $C^{\perp}$.

Theorem4.1: If C is a $(n, M, d)_2$ code over $\{0, 1\}$ with dual separation $(d_1^{\perp}, d_2^{\perp}, \ldots d_n^{\perp})$ then the codewords of C form the rows of an $OA(M, n, s, (d_1^{\perp} - 1, d_2^{\perp} - 1, \ldots, d_n^{\perp} - 1))$ with entries from {0, 1}.

Proof: Since the dual separation of C is

$(d_1^\perp, d_2^\perp, \ldots d_n^\perp)$.

$$\begin{aligned} B_{i,j}^\perp &= \frac{1}{M} \sum_{u_i \neq 0, w(\boldsymbol{u})=j} \chi_{\boldsymbol{u}}(\delta) \\ &= 0, \ for\ 1 \leq j \leq d_i^\perp - 1. \end{aligned} \tag{3}$$

Also,

$$\chi_{\boldsymbol{u}}(\delta) = \chi_{\boldsymbol{u}}(\frac{1}{M}\gamma^2) = \frac{1}{M}\chi_{\boldsymbol{u}}(\gamma)^2 \geq 0. \tag{4}$$

Therefore,

$$\chi_{\boldsymbol{u}}(\delta) = 0, \ for\ \boldsymbol{u}\ such\ that\ u_i \neq 0,\ w(\boldsymbol{u}) = j.$$

by Eq.(3), (4). Therefore, for $\boldsymbol{u}$ such that $u_i \neq 0$, $w(\boldsymbol{u}) = j$

$$\begin{aligned} \chi_{\boldsymbol{u}}(\gamma) &= \chi_{\boldsymbol{u}}(\sum_{\boldsymbol{v} \in C} \boldsymbol{z}^{\boldsymbol{v}}) \\ &= \sum_{\boldsymbol{v} \in C} (-1)^{\boldsymbol{u} \cdot \boldsymbol{v}} = 0. \end{aligned}$$

By theorem 2.1, the array of codeword of C forms an orthogonal array of strength $(d_1^\perp, d_2^\perp, \ldots, d_n^\perp)$. □

The Construction Method

The Construction Method2: Let there be two orthogonal arrays; $\overline{C_1}$ is $OA(M_1, n_1, 2, t_1)$ and $\overline{C_2}$ is $OA(M_2, n_2, 2, t_2)$, where $t_2 \leq t_1$. Let C_1 be the set of the rows of $\overline{C_1}$ and C_2 be the set of the rows of $\overline{C_2}$. Let

$$\begin{aligned} C = \{ &(c_{1,1}, c_{1,2}, \ldots, c_{1,n_1-n_{OL}+1} + c_{2,1}, \\ &\ldots, c_{1,n_1} + c_{2,n_{OL}}, c_{2,n_{OL}+1}, \ldots, c_{2,n_2}) \mid \\ &for\ \forall(c_{1,1}, c_{1,2}, \ldots, c_{1,n_1}) \in C_1, \\ &\forall(c_{2,1}, c_{2,2}, \ldots, c_{2,n_2}) \in C_2\}. \end{aligned}$$

The orthogonal array whose rows are formed by the vectors in C is $(M_1M_2) \times (n_1 + n_2 - n_{OL})$ array. Let $n_{OL} \leq t_2/2$.

Theorem4.2: The orthogonal arrays whose rows are formed by the vectors in C is $OA(M_1M_2, n_1 + n_2 - n_{OL}, 2, (r_1, r_2, \ldots, r_n))$, where

$$r_i \geq t_1 \quad (i = 1, 2, \ldots, n_1 - n_{OL}), \tag{5}$$

$$r_i \geq t_2 \quad (i = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 - n_{OL}), \tag{6}$$

$$r_i \geq t_1 + t_2 - n_{OL} \quad (i = n_1 - n_{OL} + 1, \ldots n_1). \tag{7}$$

Proof: Now, let $M = M_1M_2, n = n_1 + n_2 - n_{OL}$;

$$\gamma = \sum_{\boldsymbol{v} \in C} \boldsymbol{Z}^{\boldsymbol{v}} = \sum_{\boldsymbol{v} \in \{0,1\}^n} c_{\boldsymbol{v}} \boldsymbol{Z}^{\boldsymbol{v}},$$

$$\delta = \frac{1}{M}\gamma^2,$$

and

$$\delta^\perp = \frac{1}{M} \sum_{\boldsymbol{u} \in \{0,1\}^n} \chi_{\boldsymbol{u}}(\delta) \boldsymbol{z}^{\boldsymbol{u}}.$$

And, Let

$$C_1' = \{(a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}, 0, \ldots, 0) \in \{0,1\}^n \mid \forall(a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}) \in C_1\},$$

$$C_2' = \{(0, \ldots, 0 a_{2,1}, a_{2,2}, \ldots, a_{2,n_2}) \in \{0,1\}^n \mid \forall(a_{2,1}, a_{2,2}, \ldots, a_{2,n_2}) \in C_2\},$$

$$\gamma_1 = \sum_{\boldsymbol{v} \in C_1'} \boldsymbol{z}^{\boldsymbol{v}}, \gamma_2 = \sum_{\boldsymbol{v} \in C_2'} \boldsymbol{z}^{\boldsymbol{v}},$$

and,

$$\delta_1 = \frac{1}{M_1}\gamma_1^2, \delta_2 = \frac{1}{M_2}\gamma_2^2.$$

Then, we can describe

$$\gamma = \gamma_1 \times \gamma_2,$$

$$\delta = \frac{1}{M}\gamma^2 = \frac{1}{M_1M_2}(\gamma_1 \times \gamma_2)^2 = \delta_1 \times \delta_2.$$

Moreover,

$$\begin{aligned} \delta^\perp &= \frac{1}{M} \sum_{\boldsymbol{u} \in \{0,1\}^n} \chi_{\boldsymbol{u}}(\delta) \boldsymbol{z}^{\boldsymbol{u}} \qquad (8) \\ &= \frac{1}{M_1M_2} \sum_{\boldsymbol{u} \in \{0,1\}^n} \chi_{\boldsymbol{u}}(\delta_1 \times \delta_2) \boldsymbol{z}^{\boldsymbol{u}} \\ &= \frac{1}{M_1M_2} \sum_{\boldsymbol{u} \in \{0,1\}^n} \chi_{\boldsymbol{u}}(\delta_1)\chi_{\boldsymbol{u}}(\delta_2) \boldsymbol{z}^{\boldsymbol{u}} \\ &= \sum_{\boldsymbol{u} \in \{0,1\}^n} (\frac{1}{M_1}\chi_{\boldsymbol{u}}(\delta_1))(\frac{1}{M_2}\chi_{\boldsymbol{u}}(\delta_2)) \boldsymbol{z}^{\boldsymbol{u}}. \qquad (9) \end{aligned}$$

By Eq. (8) and (9),

$$\frac{1}{M}\chi_{\boldsymbol{u}}(\delta) = (\frac{1}{M_1}\chi_{\boldsymbol{u}}(\delta_1))(\frac{1}{M_2}\chi_{\boldsymbol{u}}(\delta_2)).$$

Therefore, for $1 \leq i \leq n, 1 \leq j \leq n$,

$$\begin{aligned} B_{i,j}^\perp &:= \frac{1}{M} \sum_{u_i \neq 0, w(\boldsymbol{u})=j} \chi_{\boldsymbol{u}}(\delta) \\ &= \sum_{u_i \neq 0, w(\boldsymbol{u})=j} (\frac{1}{M_1}\chi_{\boldsymbol{u}}(\delta_1))(\frac{1}{M_2}\chi_{\boldsymbol{u}}(\delta_2)). \end{aligned}$$

For $1 \leq i \leq n_1 - n_{OL}$,

$$B_{i,j}^\perp = 0, \ for\ 1 \leq j \leq t_1,$$

since $\chi_{\boldsymbol{u}}(\delta_1) = 0$, for $\boldsymbol{u}$ such that $u_i \neq 0, w(\boldsymbol{u}) = j$.
For $n_1 \leq i \leq n_1 + n_2 - n_{OL}$,

$$B_{i,j}^\perp = 0, \ for\ 1 \leq j \leq t_2,$$

since $\chi_{\boldsymbol{u}}(\delta_2) = 0$, for $\boldsymbol{u}$ such that $u_i \neq 0, w(\boldsymbol{u}) = j$.
For $n_1 - n_{OL} + 1 \leq i \leq n_1$,

$$B_{i,j}^\perp = 0, \ for\ 1 \leq j \leq t_1 + t_2 - n_{OL},$$

since $\chi_{\boldsymbol{u}}(\delta_1) = 0$, or, $\chi_{\boldsymbol{u}}(\delta_2) = 0$, for $\boldsymbol{u}$ such that $u_i \neq 0, w(\boldsymbol{u}) = j$.
Hence by theorem4.1, Eq.(5),(6),(7) hold. □

5 Discussion

In this section, we will show one of examples of nonlinear unequal orthogonal array constructed by the construction method2. And it will be compared to the linear unequal orthogonal array constructed by the construction method1, and the optimal equal orthogonal array [1, Table 12.1] .

Let A_1 be the orthogonal array which is constructed from $OA(24,12,2,3)$, $OA(4,3,2,2)$[1 Table 12.1] using the construciton method2. $OA(24,12,2,3)$ is a nonlinear orthogonal array. Therefore A_1 is a nonlinear unequal orthogonal array. Then A_1 is the 96×14 array. And the strength of A_1 is $(r_1, r_2, \ldots, r_{14})$, where

$$\begin{aligned} r_i &\geq 3 \ \ (i = 1, 2, \ldots, 11), \\ r_i &\geq 2 \ \ (i = 13, 14), \\ r_i &\geq 4 \ \ (i = 12). \end{aligned}$$

And let A_2 be the orthogonal array which is constructed from $OA(32,13,2,3)$, $OA(4,3,2,2)$[1 Table 12.1] using the construction method1. Both $OA(32,13,2,3)$ and $OA(4,3,2,2)$ are linear orthogonal arrays. Therefore A_2 is linear unequal orthogonal array. Then A_2 is the 128×15 array. And the strength of A_2 is $(s_1, s_2, \ldots, s_{15})$, where

$$\begin{aligned} r_i &\geq 3 \ \ (i = 1, 2, \ldots, 12), \\ r_i &\geq 2 \ \ (i = 14, 15), \\ r_i &\geq 4 \ \ (i = 13). \end{aligned}$$

And let A_3 be the orthogonal array $OA(128,14,2,4)$ [1, Table 12.1]. A_3 is optimal equal orthogonal array.

First, we compare A_1 with A_2. The number of row of A_1 is fewer than the one of A_2. Therefore A_1 can reduce more number of experiments than A_2 when the number of factors is 14.

Next, we compare A_1 with A_3 . The number of row of A_1 is fewer than the one of A_3. Therefore A_1 can reduce more number of experiments than A_3 when there are partial intersections.

And although A_2 is unequal and A_3 is equal , the number of experiments of A_2 is same as the one of A_3. But the number of experiments of A_1 is fewer than A_2, A_3.

Hence, it has been shown that there are good orthogonal arrays in orthogonal arrays constructed by the construction method2.

6 Conclusion

In this paper, we clarify the relation between nonlinear unequal orthogonal arrays and error-correcting codes. Furthermore, we extend the construction method of linear unequal orthogonal arrays, and propose one of construction methods of nonlinear unequal orthogonal arrays. And we show that there are good orthogonal arrays in orthogonal array constructed by the proposed construction method.

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