A Note on the Construction of Nonlinear Unequal Orthogonal Arrays from Error-Correcting Codes

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Abstract Orthogonal arrays have been used in the field of experimental design. Hedaya and Sloane showed the relation between orthogonal arrays and error-correcting codes[1]. And they proposed some construction methods of both linear and nonlinear orthogonal arrays from error-correcting codes. On the other hand, the paper[5] defined unequal orthogonal arrays as new class. It showed that unequal orthogonal arrays are more applicable to experimental design. Furthermore, it showed the relation between unequal orthogonal arrays and unequal error-correcting codes[3], and proposed the construction method of unequal orthogonal arrays from unequal error-correcting codes. But orthogonal arrays from this construction method are all linear. In this paper, we clarify the relation between nonlinear unequal orthogonal arrays and codes. And we propose one of construction methods of nonlinear unequal orthogonal arrays from error-correcting codes.

Key words Experimental Design, Orthogonal Arrays, Unequal Error-correcting Codes, Nonlinear Codes

1 Introduction

Experimental design has been used in many fields, for example in quality management. In experimental design, it is important to design experiments so that we can estimate the effects of factors and their interactions where the number of experiments is as few as possible.

In order to reduce the number of experiments, constructing orthogonal arrays is important[1]. Generally, there are two classes of orthogonal arrays: one is linear orthogonal arrays, the other is nonlinear orthogonal arrays.

Hedaya and Sloane showed the relation between orthogonal arrays and error-correcting codes. And they proposed some construction methods of orthogonal arrays from error-correcting codes[1]. According to their result, its relation can be divided into two types: the first is the relation between linear orthogonal arrays and linear codes, the second is the one between nonlinear orthogonal arrays and nonlinear codes. Furthermore, they proposed some construction methods of orthogonal arrays from linear and nonlinear error-correcting codes.

On the other hand, the paper[5] defined unequal orthogonal arrays as new class. From the point of view of the definition, the class with which Hedaya and Sloane dealt is equal orthogonal arrays. It showed that unequal orthogonal arrays could reduce the number of experiments when we have knowledge about effects of interactions in detail. Furthermore, it showed the relation between unequal arrays and unequal error-correcting codes[3], and proposed construction method of unequal orthogonal arrays from unequal error-correcting codes. But orthogonal arrays from this construction method are all linear.

In this paper, we clarify the relation between nonlinear unequal orthogonal arrays and codes. Furthermore, we extend the construction method of linear unequal orthogonal arrays, and propose one of construction methods of nonlinear unequal orthogonal arrays.

2 Orthogonal Arrays

2.1 Orthogonal Arrays

Definition 2.1[1] An $M \times n$ array $A$ with entries from $GF(s)$ is said to be an orthogonal array with $s$ levels and strength $t$ if every $M \times t$ subarray of $A$ contains each $t$-tuple based on $GF(s)$ exactly same times as row. We will denote such an array by $OA(M, n, s, t)$.

Example 2.1: The array in Table 1 is orthogonal array with strength 2. It is an $OA(4, 3, 2, 2)$.

We consider only the case of $s = 2$.

An orthogonal array $OA(M, n, 2, t)$ is said to be linear if the rows of $OA(M, n, 2, t)$ form a linear vector.
space. If an orthogonal array $OA(M, n, 2, t)$ is linear, $OA(M, n, 2, t)$ has a basis for the linear vector space. This basis is given in the form of $(\log_2 M) \times n$ matrix called generator matrix.

Next, we show the case that we can reduce the number of experiments when an orthogonal array is applied to an experimental design. Now we consider the case that there is a response variable of interest. And there are three factors $F_1, F_2$ and $F_3$ that might affect the response variable. Where each $F_i$ has level $F_i, 0$ and level $F_i, 1 (i = 1, 2, 3)$. Then we examine how changes in the levels of the factors affect the response variable. For example, this case is the following case. There is a ratio of defective products as a response variable. And there are three factors that might affect the response variable; the type of catalyst, machine, and material. Where each factor has two levels; catalyst0 and catalyst1, machine0 and machine1, and material0 and material1. Then we examine how changes in the levels of the factors affect the response variable.

In the above case, we can estimate all the main effects and all the interaction effects of two factors with the following experimental design:

$$\begin{align*}
(F_{1,0}F_{2,0}F_{3,0}) \\
(F_{1,0}F_{2,0}F_{3,1}) \\
(F_{1,0}F_{2,1}F_{3,0}) \\
(F_{1,0}F_{2,1}F_{3,1}) \\
(F_{1,1}F_{2,0}F_{3,0}) \\
(F_{1,1}F_{2,0}F_{3,1}) \\
(F_{1,1}F_{2,1}F_{3,0}) \\
(F_{1,1}F_{2,1}F_{3,1})
\end{align*}$$

Where the main effect of the factor $F_i$ is measurements which represent how change in the levels of the factor $F_i$ affects the response variable, and the interaction effect of the factors $F_i$ and $F_j$ is measurements which represent how changes in the level combinations of the factors $F_i$ and $F_j$ affect the response variable. Generally, an interaction effect of $k$ factors is measurements which represent how changes in the level combinations of the $k$ factors affect a response variable. And each vector of Eq.(1) corresponds to one time of experiments. For example, $(F_{1,0}F_{2,0}F_{3,0})$ correspond to the experiment whose level combination is $F_{1,0}, F_{2,0}, F_{3,0}$. This experimental design contains all level combinations as the vectors. This design such as Eq.(1) is said to be complete design.

Further, we suppose that we know that there is no interaction effect of two factors by experiences. Then we can estimate all the main effect of factors with the following experimental design:

$$\begin{align*}
(F_{1,0}F_{2,0}F_{3,0}) \\
(F_{1,0}F_{2,1}F_{3,1}) \\
(F_{1,1}F_{2,0}F_{3,1}) \\
(F_{1,1}F_{2,1}F_{3,0})
\end{align*}$$

This experimental design is made using the orthogonal array $OA(4, 3, 2, 2)$ we show in Table 1. Where, each row of the $OA(4, 3, 2, 2)$ corresponds to the vectors in Eq.(2). For example, 000 which is the first row of the $OA(4, 3, 2, 2)$ correspond to $(F_{1,0}F_{2,0}F_{3,0})$. Then we can reduce the number of experiments using the design that is Eq.(2).

As the above case, when we have knowledge of interaction effects, we can reduce the number of experiments using an orthogonal array. Generally, when an orthogonal array $OA(M, n, 2, t)$ is applied to experimental design, the number of experiments is $M$, and we can estimate interaction effects of at most $\frac{\log_2 M}{2}$ factors.

Next we describe a necessary and sufficient condition for an array to be an orthogonal array.

**Theorem 2.1** An $M \times n$ array $A$ with 0, 1 entries is an $OA(M, n, 2, t)$ if and only if

$$\sum_{u \in \text{row} \in A} (-1)^{u \cdot v} = 0,$$

for all 0, 1 vectors $v$ containing $w$ 1's, for all $w$ in the range $1 \leq w \leq t$, where the sum is over all rows $u$ of $A$.

### 2.2 Orthogonal Arrays From Codes

#### 2.2.1 Linear Orthogonal Arrays from Linear Codes

Let $w(u)$ be the Hamming weight of a vector $u = (u_1, u_2, \ldots, u_n)$. An error-correcting code or simply code is any collection $C$ of vectors in $GF(s)^n$. The vectors in $C$ are called codewords. Let $\text{dist}(u, v)$ be the Hamming distance between two vectors $u, v$. We define the minimal distance $d$ of a code $C$ to be the minimal distance between distinct codewords:

$$d = \min_{u, v \in C, u \neq v} \text{dist}(u, v).$$

If $C$ contains $M$ codewords, then we say that it is a code of length $n$, size $M$ and minimal distance $d$ over $GF(s)$ or simply $(n, M, d)_s$ code. We consider the case of $s = 2$ as orthogonal arrays.

$C$ is said to be linear if $C$ is a linear vector subspace. If $C$ is linear, $C$ has the dual code $C^\perp$. Let $d^\perp$ be the minimal distance of $C^\perp$. Then $d^\perp$ is said to be a dual distance of $C$.

**Theorem 2.2** If $C$ is a $(n, M, d)_2$ linear code over $(0, 1)$ with dual distance $d^\perp$ then the codewords of $C$ form the rows of an $OA(M, n, 2, d^\perp - 1)$ with entries from $(0, 1)$. 

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2.2.2 Nonlinear Orthogonal Arrays from Nonlinear Codes

The Group Algebra

we are going to describe binary vectors of length \( n \) by polynomials in \( z_1, z_2, \ldots, z_n \). For example, \( 100 \ldots 0 \) will be represented by \( z_1 \), \( 101 \ldots 0 \) by \( z_1 z_3 \) and so on. In general \( v = v_1 v_2 \ldots v_n \) is represented by \( z_1^{v_1} z_2^{v_2} \ldots z_n^{v_n} \), which we abbreviate \( z^v \). We make the convention that \( z_i^0 = 1 \) for all \( i \). This makes the set of all \( z^v \) into a multiplicative group denoted by \( G \). Thus \( \{0,1\}^n \) and \( G \) are isomorphic groups. With addition in \( \{0,1\}^n \)

\[
\begin{align*}
v + w &= (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) \\
&= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n),
\end{align*}
\]

corresponding to multiplication in \( G \):

\[
\begin{align*}
z^{v}z^{w} &= z_1^{v_1}z_2^{v_2} \cdots z_n^{v_n} \\
&= z_1^{w_1}z_2^{w_2} \cdots z_n^{w_n} = z^{v+w}.
\end{align*}
\]

Definition 2.2:[2] The group algebra \( QG \) of \( G \) over the rational numbers \( Q \) consists of all formal sums

\[
\sum_{v \in \{0,1\}^n} a_v z^v, \quad a_v \in Q, \quad z^v \in G.
\]

Addition and multiplication of elements of \( QG \) are defined in the natural way by

\[
\begin{align*}
\sum_{v \in \{0,1\}^n} a_v z^v + \sum_{v \in \{0,1\}^n} b_v z^v &= \sum_{v \in \{0,1\}^n} (a_v + b_v) z^v, \\
r \sum_{v \in \{0,1\}^n} a_v z^v &= \sum_{v \in \{0,1\}^n} r a_v z^v, \quad r \in Q,
\end{align*}
\]

and

\[
\sum_{v \in \{0,1\}^n} a_v z^v \cdot \sum_{v \in \{0,1\}^n} b_w z^w = \sum_{v, w \in \{0,1\}^n} a_v b_w z^{v+w}.
\]

To each \( u \in \{0,1\} \), we associate the mapping \( \chi_u \) from \( G \) to the rational numbers given by

\[
\chi_u(z^v) = (-1)^{u \cdot v},
\]

where \( u \cdot v \) is the scalar product of the vectors \( u, v \) over \( Q \). \( \chi_u \) is called a character of \( G \). \( \chi_u \) is extended to act on \( QG \) by linearity.

\[
\chi_u \left( \sum_{v \in \{0,1\}^n} a_v z^v \right) = \sum_{v \in \{0,1\}^n} a_v \chi_u(z^v) = \sum_{v \in \{0,1\}^n} (-1)^{u \cdot v} a_v.
\]

Let

\[
\gamma = \sum_{v \in \{0,1\}^n} c_v z^v,
\]

be an arbitrary element of the group algebra \( QG \) with the property that

\[
M = \sum_{v \in \{0,1\}^n} c_v \neq 0.
\]

We call the \((n+1)\)-tuple \( \{A_0, A_1, \ldots, A_n\} \), where

\[
A_i = \sum_{w(u) = i} c_v,
\]

the weight distribution of \( \gamma \).

Definition 2.3:[2] The transform of \( \gamma \) is the element \( \gamma^t \) of \( QG \) given by

\[
\gamma^t = \frac{1}{M} \sum_{u \in \{0,1\}^n} \chi_u(\gamma) z^u.
\]

Nonlinear Orthogonal Arrays from Codes

Now let \( C \) be a linear or nonlinear \((n, M, d)\) code. \( C \) is described by the element

\[
\gamma = \sum_{v \in C} z^v,
\]

of \( QG \). Let \( \delta = \frac{1}{M} \gamma^2 \), and \( \delta^t \) is the transform of \( \delta \). The weight distribution of \( \delta^t \) is \( \{B_0^t, B_1^t, \ldots, B_n^t\} \), where

\[
B_i^t = \frac{1}{M} \sum_{u \in \{0,1\}^n} \chi_u(\delta).
\]

Definition 2.4:[2] The dual distance \( d^t \) of a code \( C \) is defined by \( B_i^t = 0 \) for \( 1 \leq i \leq d^t - 1 \), \( B_n^t \neq 0 \), if \( C \) is linear, \( d^t \) is the minimum distance of \( C^t \).

Theorem 2.3:[1][2] If \( C \) is a \((n, M, d)\) code over \( \{0,1\} \) with dual distance \( d^t \), then the codewords of \( C \) form the rows of an \( OA(M, n, 2, d^t - 1) \) with entries from \( \{0,1\} \).

3 Unequal Orthogonal Arrays

3.1 Unequal Orthogonal Arrays

Definition 3.1: An \( M \times n \) array \( A \) with \( 0,1 \) entries is said to be an unequal orthogonal array with \( 2 \) levels and strength \( r = (r_1, r_2, \ldots, r_n) \) if every \( M \times r_i \) subarray of \( A \), which contain \( ith \) column of \( A \), contains each \( r_i \)-tuple based on \( \{0,1\} \) exactly the same times as row. We will denote such an array by \( OA(M, n, 2, r) \).

When \( OA(M, n, 2, (r_1, r_2, \ldots, r_n)) \) is applied to experimental design, we can estimate the effects of interactions of at most \([\frac{n}{2}]\) factors which contains \( ith \) factor. It was shown that there are cases that unequal array reduce more numbers of experiments than equal orthogonal arrays[9].
For example, this case is the following case. Here, let $F_1 \times F_2$ be the interaction of $F_1$ and $F_2$. We suppose that there are three factors $F_1, F_2$ and $F_3$. And we know that there are $F_1 \times F_2$ and $F_1 \times F_3$. When an equal orthogonal array $OA(M_1, 3, 2, 4)$ is used, we can estimate not only $F_1 \times F_2, F_1 \times F_3$ but $F_2 \times F_3$, although we need not estimate $F_2 \times F_3$. On the other hand, when an unequal orthogonal array $OA(M_2, 3, 2, (4, 2, 2))$ is used, we can not estimate $F_2 \times F_3$. Therefore unequal orthogonal array reduces the number of experiments.

Next we describe a necessary and sufficient condition for an array to be an unequal orthogonal array.

**Theorem 3.1**: An $M \times n$ array $A$ with 0, 1 entries is an $OA(M, n, 2, (r_1, r_2, \ldots, r_n))$ if and only if

$$
\sum_{u=v=A} (-1)^{u \cdot v} = 0,
$$

for all 0, 1 vectors $v = (v_1, v_2, \ldots, v_n)$ such that $v_i \neq 0$ and $w(v) = \sum w$ for all $w$ in the range $1 \leq w \leq r_i$, where the sum is over all rows $w$ of $A$.

### 3.2 Unequal Orthogonal Arrays from Code

**3.2.1 Linear Unequal Orthogonal Arrays from Codes**

The separation $(d_1, d_2, \ldots, d_n)$ of linear code $C$ is defined by

$$
d_i = \min \{ \text{dist}(u, v) \mid u = (u_1, u_2, \ldots, u_n), \quad v = (v_1, v_2, \ldots, v_n), \quad u, v \in C, u \neq v, \}
$$

for $i = 1, 2, \ldots, n$.

Let $(d_1^L, d_2^L, \ldots, d_n^L)$ be the separation of $C^L$ which is the dual code of $C$. Then $(d_1^L, d_2^L, \ldots, d_n^L)$ is said to be a dual separation of $C$.

**Theorem 3.2**: If $C$ is a $(n, M, d)$ linear code over $\{0, 1\}$ with dual separation $(d_1^L, d_2^L, \ldots, d_n^L)$, then the code-words of $C$ form the row of an $OA(M, n, 2, (d_1^L - 1, d_2^L - 1, \ldots, d_n^L - 1))$ with entries from $\{0, 1\}$.

The **Construction Method**

Now we show the construction method of orthogonal array. This construction method is the method that is applied to the construction method of unequal error-correcting code[3] to.

The **Construction Method1**: Let there be two generator matrix of orthogonal arrays; $G_1$ is the generator matrix of a linear orthogonal array $OA(M_1, n_1, 2, t_1)$, and $G_2$ is the one of a linear orthogonal array $OA(M_2, n_2, 2, t_2)$, where $t_2 \leq t_1$. Let $G_1$ and $G_2$ be joined as submatrices of $G$ where $G_1$ and $G_2$

![Figure 1: The construction method of linear orthogonal array](image)

overlap, as shown in Figure 1. The orthogonal array for which $G$ is generator matrix is $(M_1M_2 \times (n_1 + n_2 - n_{0L}))$ array. Let $n_{0L} \leq t_2/2$.

**Theorem 3.3**: The orthogonal arrays for which $G$ is generator matrix is $OA(M_1M_2, n_1 + n_2 - n_{0L}, 2, (t_1, t_2, \ldots, r_n))$, where

$$
\begin{align*}
\gamma_i & \geq t_1 \quad (i = 1, 2, \ldots, n_1), \\
\gamma_i & \geq t_2 \quad (i = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 + n_{0L}), \\
\gamma_i & \geq t_1 + t_2 - n_{0L} \quad (i = n_1 + n_{0L} + 1, \ldots, n_1).
\end{align*}
$$

### 4 Nonlinear Unequal Orthogonal Arrays

**Nonlinear Unequal Orthogonal Arrays from Codes**

Let $C$ be a linear or nonlinear $(n, M, d)$ code and

$$
\gamma = \sum_{v \in C} \gamma_v.
$$

Let $\delta = \frac{1}{d^L} \gamma^2$ and $\delta^L$ is the transform of $\delta$. Now $(B_{i_1, i_2}, B_{i_2, i_3}, \ldots, B_{i_n, i_1})$, for $i = 1, 2, \ldots, n$ is defined by

$$
B_{i,j} = \frac{1}{M} \sum_{u \neq 0, w(u) = j} w(u, \delta).
$$

**Definition 4.1**: The dual separation $(d_1^L, d_2^L, \ldots, d_n^L)$ of $C$ is defined by

$$
B_{i,j} = 0, \quad \text{for} \quad 0 \leq j \leq d_i^L - 1,
$$

$$
B_{d_i^L} \neq 0.
$$

If $C$ is linear, $(d_1^L, d_2^L, \ldots, d_n^L)$ is the separation of $C^L$.

**Theorem 4.1**: If $C$ is a $(n, M, d)$ code over $\{0, 1\}$ with dual separation $(d_1^L, d_2^L, \ldots, d_n^L)$ then the code-words of $C$ form the rows of an $OA(M, n, 2, (d_1^L - 1, d_2^L - 1, \ldots, d_n^L - 1)$ with entries from $\{0, 1\}$.

**Proof**: Since the dual separation of $C$ is
\[(d_1^d, d_2^d, \ldots, d_n^d).\]
\[
B^d_{i,j} = \frac{1}{M} \sum_{u, v, w(u) = j} \chi_u(d) = 0, \text{ for } 1 \leq j \leq d_1^d - 1.
\] (3)

Also,
\[
\chi_u(d) = \chi_u \left( \frac{1}{M} \gamma^2 \right) = \frac{1}{M} \chi_u(\gamma)^2 \geq 0.
\] (4)

Therefore,
\[
\chi_u(\delta) = 0, \text{ for } u \text{ such that } u_i \neq 0, w(u) = j.
\]

by Eq.(3), (4). Therefore, for \(u\) such that \(u_i \neq 0, w(u) = j\)
\[
\chi_u(\gamma) = \chi_u \left( \sum_{v \in C} z^v \right) = \sum_{v \in C} (-1)^{u \cdot v} = 0.
\]

By theorem 2.1, the array of codeword of \(C\) forms an orthogonal array of strength \((d_1^d, d_2^d, \ldots, d_n^d). \) \(\Box\)

The Construction Method

The Construction Method2: Let there be two orthogonal arrays; \(C_1\) is \(OA(M_1, n_1, 2, t_1)\) and \(C_2\) is \(OA(M_2, n_2, 2, t_2)\), where \(t_1 \leq t_1 \leq t_2\). Let \(C_1\) be the set of the rows of \(C_1\) and \(C_2\) be the set of the rows of \(C_2\). Let
\[
C = \{(c_{1,1}, c_{1,2}, \ldots, c_{1,n_1-n_{nL}+1}^1, \ldots, c_{2,n_2}, c_{2,n_2-1}, \ldots, c_{2,n_2}) | \forall (c_{1,1}, c_{1,2}, \ldots, c_{1,n_1}) \in C_1, \forall (c_{2,1}, c_{2,2}, \ldots, c_{2,n_2}) \in C_2\}.
\]

The orthogonal array whose rows are formed by the vectors in \(C\) is \((M_1 M_2) \times (n_1 + n_2 - n_{nL})\) array. Let \(n_{nL} \leq t_2/2\).

Theorem 4.2: The orthogonal arrays whose rows are formed by the vectors in \(C\) is \(OA(M_1 M_2, n_1 + n_2 - n_{nL}, 2, (t_1, t_2, \ldots, r_n))\), where
\[
r_i \geq t_i \quad (i = 1, 2, \ldots, n_1 - n_{nL}). \quad (5)
\]
\[
r_i \geq t_2 \quad (i = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2 - n_{nL}). \quad (6)
\]
\[
r_i \geq t_1 + t_2 - n_{nL} \quad (i = n_1 - n_{nL} + 1, \ldots, n_1). \quad (7)
\]

Proof: Now, let \(M = M_1 M_2, n = n_1 + n_2 - n_{nL}\).
\[
\gamma = \sum_{v \in C} Z^v = \sum_{v \in \{0,1\}^n} c_v Z^v,
\]
and
\[
\delta = \frac{1}{M} \gamma^2,
\]
and
\[
\delta^2 = \frac{1}{M} \sum_{u \in \{0,1\}^n} \chi_u(\delta) z^u.
\]

And, let
\[
C_1 = \{(a_1, a_2, \ldots, a_{n_1}, 0, \ldots, 0) \in \{0,1\}^n | \forall (a_1, a_2, \ldots, a_{n_1}) \in C_1\}.
\]
\[
C_2 = \{(0, \ldots, 0, a_{n_2}, a_{n_2+1}, \ldots, a_{n_2}) \in \{0,1\}^n | \forall (a_{n_2}, a_{n_2+1}, \ldots, a_{n_2}) \in C_2\},
\]
\[
\gamma_1 = \sum_{v \in C_1^2} z^v, \gamma_2 = \sum_{v \in C_2^2} z^v,
\]
and
\[
\delta_1 = \frac{1}{M_1} \gamma_1^2, \delta_2 = \frac{1}{M_2} \gamma_2^2.
\]

Then, we can describe
\[
\gamma = \gamma_1 \times \gamma_2,
\]
\[
\delta = \frac{1}{M} \frac{1}{M_1 M_2} (\gamma_1 \times \gamma_2)^2 = \delta_1 \times \delta_2.
\]

Moreover,
\[
\delta^2 = \frac{1}{M} \sum_{u \in \{0,1\}^n} \chi_u(\delta) z^u
\]
\[
= \frac{1}{M_1 M_2} \sum_{u \in \{0,1\}^n} \chi_u(\delta_1 \times \delta_2) z^u
\]
\[
= \frac{1}{M_1 M_2} \sum_{u \in \{0,1\}^n} \chi_u(\delta_1) \chi_u(\delta_2) z^u
\]
\[
= \sum_{u \in \{0,1\}^n} \left( \frac{1}{M_1} \chi_u(\delta_1) \right) \left( \frac{1}{M_2} \chi_u(\delta_2) \right) z^u. \quad (9)
\]

By Eq. (8) and (9),
\[
\frac{1}{M} \chi_u(\delta) = \left( \frac{1}{M_1} \chi_u(\delta_1) \right) \left( \frac{1}{M_2} \chi_u(\delta_2) \right).
\]

Therefore, for \(1 \leq i \leq n_1 - n_{nL}, \)
\[
B_{i,j}^d := \frac{1}{M} \sum_{u, v, w(u) = j} \chi_u(\delta)
\]
\[
= \sum_{u, \neq 0, w(u) = j} \left( \frac{1}{M_1} \chi_u(\delta_1) \right) \left( \frac{1}{M_2} \chi_u(\delta_2) \right).
\]

For \(1 \leq i \leq n_1 - n_{nL}, \)
\[
B_{i,j}^d = 0, \text{ for } 1 \leq j \leq t_1,
\]
\[
\text{since } \chi_u(\delta_1) = 0, \text{ for } u \text{ such that } u_i \neq 0, w(u) = j.
\]
\[
\text{For } n_1 \leq i \leq n_1 + n_2 - n_{nL}, \)
\[
B_{i,j}^d = 0, \text{ for } 1 \leq j \leq t_2,
\]
\[
\text{since } \chi_u(\delta_2) = 0, \text{ for } u \text{ such that } u_i \neq 0, w(u) = j.
\]
\[
\text{For } n_1 - n_{nL} + 1 \leq i \leq n_1,
\]
\[
B_{i,j}^d = 0, \text{ for } 1 \leq j \leq t_1 + t_2 - n_{nL},
\]
\[
\text{since } \chi_u(\delta_1) = 0, \text{ or } \chi_u(\delta_2) = 0, \text{ for } u \text{ such that } u_i \neq 0, w(u) = j.
\]

Hence by theorem 4.1, Eq.(5),(8),(7) hold. \(\Box\)
5 Discussion

In this section, we will show one of examples of nonlinear unequal orthogonal array constructed by the construction method. And it will be compared to the linear unequal orthogonal array constructed by the construction method, and the optimal equal orthogonal array [1, Table 12.1].

Let $A_1$ be the orthogonal array which is constructed from $OA(24, 12, 2, 3)$, $OA(4, 3, 2, 2)$ [1 Table 12.1] using the construction method. $OA(24, 12, 2, 3)$ is a nonlinear orthogonal array. Therefore $A_1$ is a nonlinear unequal orthogonal array. Then $A_1$ is the $96 \times 14$ array. And the strength of $A_1$ is $(r_1, r_2, \ldots, r_{14})$, where

$$r_i \geq 3 \quad (i = 1, 2, \ldots, 11),$$
$$r_i \geq 2 \quad (i = 13, 14),$$
$$r_i \geq 4 \quad (i = 12).$$

And let $A_2$ be the orthogonal array which is constructed from $OA(32, 13, 2, 3)$, $OA(4, 3, 2, 2)$ [1 Table 12.1] using the construction method. Both $OA(32, 13, 2, 3)$ and $OA(4, 3, 2, 2)$ are linear orthogonal arrays. Therefore $A_2$ is linear unequal orthogonal array. Then $A_2$ is the $128 \times 15$ array. And the strength of $A_2$ is $(s_1, s_2, \ldots, s_{15})$, where

$$r_i \geq 3 \quad (i = 1, 2, \ldots, 12),$$
$$r_i \geq 2 \quad (i = 14, 15),$$
$$r_i \geq 4 \quad (i = 13).$$

And let $A_3$ be the orthogonal array $OA(128, 14, 2, 4)$ [1, Table 12.1]. $A_3$ is optimal equal orthogonal array.

First, we compare $A_1$ with $A_2$. The number of row of $A_1$ is fewer than the one of $A_2$. Therefore $A_1$ can reduce more number of experiments than $A_2$ when the number of factors is 14.

Next, we compare $A_1$ with $A_3$. The number of row of $A_1$ is fewer than the one of $A_3$. Therefore $A_1$ can reduce more number of experiments than $A_3$ when there are partial intersections.

And although $A_2$ is unequal and $A_3$ is equal, the number of experiments of $A_2$ is same as the one of $A_3$. But the number of experiments of $A_1$ is fewer than $A_2, A_3$.

Hence, it has been shown that there are good orthogonal arrays in orthogonal arrays constructed by the construction method.

6 Conclusion

In this paper, we clarify the relation between nonlinear unequal orthogonal arrays and error-correcting codes. Furthermore, we extend the construction method of linear unequal orthogonal arrays, and propose one of construction methods of nonlinear unequal orthogonal arrays. And we show that there are good orthogonal arrays in orthogonal array constructed by the proposed construction method.

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