Information Theoretic Consideration of Multi-Layered Inventory Process

***Masayuki Goto, *Tadayuki Masui, **Shigeichi Hirasawa, ***Nobuhiko Tawara**

*Fac. of Environmental and Information Studies, Muasahi Institute of Technology, JAPAN

**School of Science and Engineering, Waseda University, JAPAN

***Dep. of Engineering, Musashi Institute of Technology, JAPAN

Abstract

In this paper, the models of multi-layered inventory process are proposed besed on information theory. Recently, the physical distribution is one of the main important activities of many companies. The physical distribution can be regarded as the multi-layered inventory process represented by stock points and paths between those. On the other hand, the effectiveness of supply chain management between companies has been discussed and new value chain models named CTO (Configure to Order) or BTO (Build to Order), and so on appeared in practice. The essential point of these value chain models may be the generality of inventory in each stock points. In this paper, we discuss value chain models expressing the generality of inventories in each stock points, and analyze the properties of stock from the viewpoints of information theory. At first, we define the generality of inventory in each stock point using the entropy and evaluate the generality of the whole value chain process. Secondly, the relation of the optimization between the value chain and information compression in information theory is stated and we show an optimization algorithm to acquire an optimal value chain structure. At last, we define the divergence level of inventory and formulate the optimization problem. The relation between the formulation in this paper and some results in Information Theory is clarified.

Keywords:

Supply Chain, Information Theory, Logistics, Value Chain, Entropy, Business Model

1 INTRODUCTION

Recently, the research and approach to developing business model using Information Technology (IT) and its application for industrial managemtnt have been remarkable. In this knowledge era, the widespread adoption of IT, particularly in business activities, is transforming the business paradigm, redefining the approach to business. The perception, however, that IT is but a means to access and transmit information is quickly being outdated as technology and network systems evolve around the world. The company which can appropriately use the value of IT would survive in the future.

The physical distribution and supply chain are one of the most important business processes in this knowledge era [1]. The effective distribution systems have been constructed in many corporations by using IT. In supply chain, the collaboration and the coordination equipment between distributed customers, operation and suppliers are increasingly demanded and may be realized by help of IT. In fact, many corporations already apply IT on assisting logistics and supply chain [1]-[6].

On the other hand, the successful cases of the DELL Co. brought us new aspect such as the Configure To Order (CTO) business model is sometimes effective for the recent market. This model can be obviously explained by division of a production and distribution process into two stages, such as, the forecast investment stage and order achievement stage. In the forecast investment stage, the buyer of a product in process has not been decided yet. After a customer gives an order with his or her customizing to the company, the ordered product will proceed to the order achievement stage. In this stage, the process to finish production is executed. From the viewpoint of difference of turning point between two stages, the Engineer to Order (ETO) and the Build to Order (BTO) business models were well known as similar models with the CTO model. The essence of the CTO model is not to finish the final assembly of the product before the customer's order. In the DELL Co., the customizing and final assembly are done only after the customer orders, and the products are sent to the customer promptly within the short Lead Time (LT). In this business model, instead customers select from complete products in a shop, he can get his own ordered product. Because the user of a product has not decided before the order achievement stage, it or its parts can be diverted to other purpose if the demand attenuates. Therefore, the unsold stuff risk in sales is reduced. If the parts and Works-In –Process (WIT), i.e. inventories, can be diverted for other products, it is not necessary to abandon those, and the possibility of leading to huge abandonment cont for the enterprise has been left. In this paper, the possibility of an inventory use is called "*Generality of Inventory*".

The purpose of this paper is to propose and analyze the multi-layered inventory process model to clarify the role of the generality of the inventory from the viewpoint of Information Theory [7]-[9]. The inventory tree model is introduced and it is formulated by probability theory. Then, the entropy criterion can be defined and the minimization problem of multi-layered distribution process is introduced. It is shown that all such viewpoints relate closely to the information theory.

2 MULTI-LAYERD INVENTORY PROCESS AND TREE STRUCTURE

In Figure.1, an example of the multi-layered inventory process model is shown. The number of the layers is 4 in this case. It can be regarded as a value chain model. Each node in the tree model is representing *a inventory point*. It has not been decided for the inventories in an intermediate node to be distributed to which demand point, that is, the intermediate inventories have uncertainty in the meaning of destination and consumption point. Each leaf in the tree is meaning a *Shop*. Of course, each shop has stock to correspond to change of demand.

This tree model representing the multi-layered inventory process can be regarded as an analogy of tree probability model which is often useful in Information Theory.

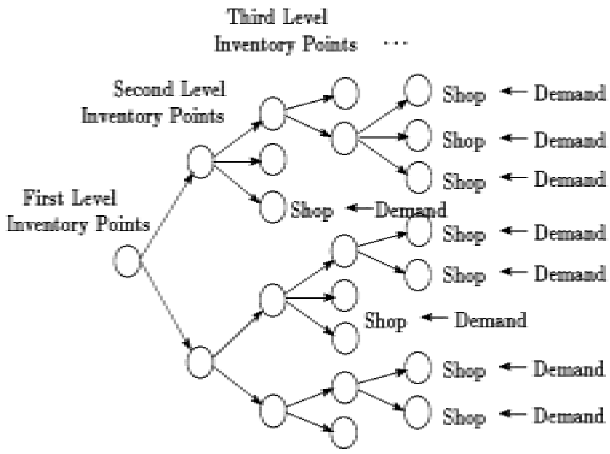

Figure 1: An example of multi-layered inventory process

3 GENERALITY OF INVENTORY AND ENTROPY

3.1 Preliminaries

In this paper, let the multi-layered inventory process model be redefined as a tree probability model. By representing it by the tree structure, the relation to the results in the area of information theory is clarified. To do this, the weights meaning the amount ratios of intermediate inventories standardized by the all aggregated demand are introduced. A summation of all weights in each layers equals to 1.

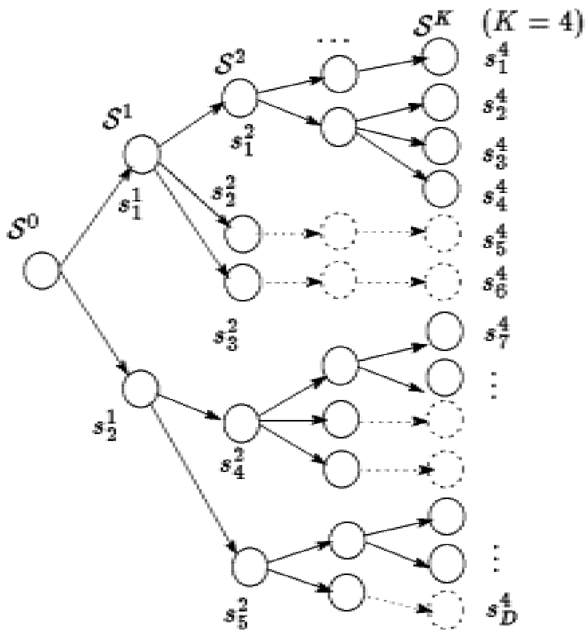

Figure 2: Notations of multi-layered inventory process

Let the root node of the tree be S^0. The set of all nodes in the k-th level inventory points is written as S^k and elements in S^k are written as $s^k{}_j$. Let the random variable over S^k be X_k and the probabilities of nodes are defined by

$$p_j^k = \Pr\{X_k = s_j^k\} = \frac{d_j^k}{\sum_i d_i^k} , \tag{1}$$

$$d_j^k = \sum_{s_j^K \in D_j^k} d_j^K , \tag{2}$$

where D_j^k is a set of all leaf node for which s_j^k is an ancestor node and d_j^k represents the demand of the leaf node s_j^k. For all k, the equation

$$\sum_j d_j^k = \sum_i d_i^K \tag{3}$$

holds. A realization value of X_k is written by x_k. Thus $x_k \in S_k$.

3.2 Generality of Inventory

Then, the generality of intermediate inventory in this paper is defined as follows:

Definition 1: For all $k \le K$, *the generality of inventory* at a node s_j^k is defined by the entropy

$$H\left(X_K \middle| s_j^k\right) = -\sum_i \left(\frac{p_i^K}{p_j^k}\right) \log\left(\frac{p_i^K}{p_j^k}\right) \tag{4}$$

given s_j^k. The generality of *k*-th level inventories is defined by the conditional entropy

$$H\left(X_K \middle| X_k\right) = \sum_j p_j^k H\left(X_K \middle| s_j^k\right) \tag{5}$$

□

As Definition 1, the generality of inventory can be regarded as its uncertainty of destination.

One of the most important decision making in logistics and supply chain is to decide where the stock base is set up in practice. The problem of decision of stock points can be reduced to decision making of tree structure in multi-layered inventory process model. Usually, stock points should be set from the many viewpoints of Lead Time (LT), cost and efficiency of transportation, locality and geographical features, and so on. Additionally, we can take the strategy such that the stock points are selected by the criterion of *generality of inventory*. If the generality of inventory is wanted to be maintained, the strategy to maximize total generality of tree model

$$L_1 = \sum_{k=1}^{K-1} H\left(X_K \middle| X_k\right) \tag{6}$$

can be applied.

In this section, to consider the problem of maximization of the criteria (6), let give a list of conditions.

Condition 1:

1. The depth of tree, *K*, is fixed.
2. The number of each leaf node (shop) and its probability are previously given. That is, the probability distribution of X_K is known.

3. The number of intermediate nodes from level 2 to level K-1 is changeable.
4. For $k = 1, 2, \cdots, K-2$, the link of each node in k-th level to child node is changeable. □

In the above assumption, the shops already exist but rooting to each destination and stock points have not been decided yet. We can formulate the entropy maximization problem to decide the tree structure from the viewpoint of generality of intermediate inventories.

Then the next theorem holds.

Theorem 1 :

Assume Condition 1. Then, the maximization of

$$L_1 = \sum_{k=1}^{K-1} H\left(X_K \middle| X_k\right) \tag{7}$$

with respect to tree structures is equivalent to the minimization of

$$\sum_{k=1}^{K-1} H\left(X_k\right) \tag{8}$$

(Proof)

Using $H\left(X_k \middle| X_K\right) = 0$, the criteria (7) can be transformed as

$$\sum_{k=1}^{K-1} H\left(X_K \middle| X_k\right) = \sum_{k=1}^{K-1} \left\{H\left(X_K\right) - H\left(X_k\right)\right\}$$

$$= (K-1) H\left(X_K\right) - \sum_{k=1}^{K-1} H\left(X_k\right)$$

The proof is complete. □

Theorem 1 says that the peculiar tree structure such that the number of intermediate nodes in $2, \cdots, (K-2)$-th levels is 1 and It diverges to all the leaves at $(K-1)$*-th* level is optimal. An inventory tree with structure like broom maximizes the generality of inventory criterion (7).

In practice, the logistics model represented by this peculiar tree structure cannot be taken. There are many objective functions to maximize at the same time. Therefore, a criterion of generality should be introduced multipurpose optimization. If some alternatives are given, we can calculate the generality of inventory for each tree model and compare with each other to decide preferable logistics model.

4 PROBLEM OF MINIMIZATION DISTRIBUTION PROCESS AND RELATION WITH CODING THEOREM

4.1 Preliminaries

In this section, let consider the problem of minimization of distribution process length when the number of nodes in each level is limited but the depth of tree model, K, is changeable. The distribution process length depends on Lead Time in practice and minimization the length is significant in real situations when the points of inventory base are given. Usually, it needs a large cost to set up the large distribution center and warehouse. Therefore, the number of nodes in high layer level may be smaller than that in low layer level.

Instead the number of nodes in each level is limited, let us set a restriction for number of divergence in each node. Let V_k be the maximum number of divergence in a node in the k-th level. In Figure 3, an example of tree is shown in the case V_2=2, V_3=3, and V_4=4.

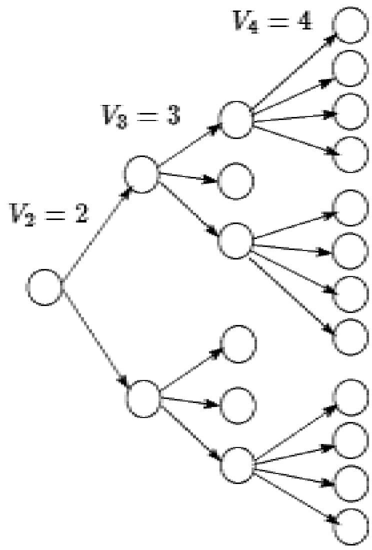

Figure 3: Example of limited tree structure

Usually, if the scale of distribution center and warehouse is small, then many bases can be set up. Therefore, the inequality

$$V_1 \le V_2 \le V_3 \cdots \tag{9}$$

can be naturally assumed. Moreover, it is assumed that the number of shops D is given in this section. A shop is meaning a demand point, so it is given and cannot be changed by distribution traders.

Here, let l_i be *a process length* of the i-th demand point represented by a leaf node of the tree. That is, the depth of the i-th leaf node is l_i.

By the above settings, the problem of decision making of a tree structure is very similar to the source coding problem in Information Theory. The difference is the number of divergence in each node. In Information Theory, a link or branch is meaning *a code symbol in alphabet* and then the size of alphabet is fixed. In problems of the multi-layered inventory process, the number of divergence of node is changeable in practice. The setting defined in this paper is more general than the coding problem.

Then, the following important theorem can be obtained.

Theorem 2 (Generalized Craft's inequality) :

Let the maximum number of divergence of node in the level k be V_k. For D demand points, the necessary and sufficient condition to exist some distribution process (tree model) such that route lengths are $l_1, l_2, \cdots, l_D$, is given by

$$\frac{1}{\prod_{i=1}^{l_1} V_i} + \frac{1}{\prod_{i=1}^{l_2} V_i} + \cdots \cdots + \frac{1}{\prod_{i=1}^{l_D} V_i} \le 1 \tag{10}$$

□

The inequality (7) reduces to Craft's inequality [7]-[9] when V_k is constant. If $V = V_k$ for all *k*, then

$$V^{-l_1} + V^{-l_2} + \cdots + V^{-l_D} \leq 1 \qquad (11)$$

Though Theorem 2 is of course important in this problem, it may be more significant in Information Theory. Figure 4 shows the essence the generalized Craft's inequality. In the settings in Information Theory, the number of divergence in each node is fixed to some value depending on the alphabet size used for source coding.

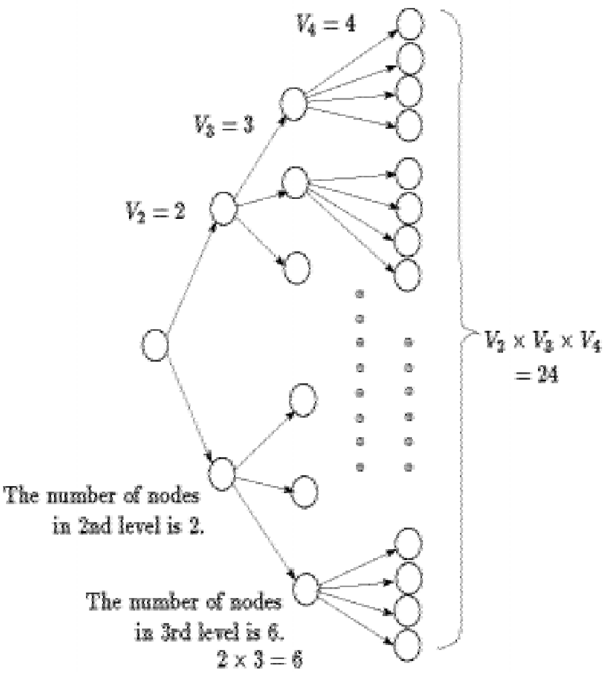

Figure 4: Figure meaning of generalized Craft's inequality

In the following subsections, the min-max process length problem and the minimum average process length problem are formulated. In these problems, the solution of optimal tree structure will be searched within trees satisfying (10).

4.2 Formulation of Min-Max Process Length Problem

Here, let consider the problem to minimize a maximum length in $l_1, l_2, \cdots, l_D$. That is, this is a strategy to optimize the worst case of distribution root length. Without loss of generality, we assume that the probability masses are ordered, so that

$$p_1 \leq p_2 \leq \cdots \leq p_D$$

In this problem, the optimal algorithm to acquire the optimal tree structure is obviously given as follows:

Optimal Algorithm 1:

1. (Step1) Find a natural number *n* satisfying

   $$\prod_{i=1}^{n} V_i \leq D \leq \prod_{i=1}^{n+1} V_i$$

   and set *j*=0, *k*=1,

   $$D' = \left| D - \prod_{i=1}^{n} V_i \right|$$

2. (Step2) For $N = \min\{D' + k, V_{n+1}\}$ and leafs with probabilities $p_{j+1}, p_{j+2}, \cdots, p_{j+N}$, make a node and link this node as a parent of these leafs. If $N = D' + k$, then go to Step5.
3. (Step3) If $N = D' + k$, then go to Step5. Otherwise, set *j*=j+*N*, *k*=*k*+1, $D' = D' - V_{n+1}$, and go to Step3.
4. (Step4) The remaining leafs p_{j+n+1}, p_{j+n+2}, $\cdots, p_D$ are allocated to the leaf nodes in n-depth complete tree respectively.
5. (Step5) Let the leafs which are not allocated in Step5 be the intermediate nodes and correspond it to the node made in Step3. Let all p_{j+1}, p_{j+2}, $\cdots, p_{j+N}$ correspond to leafs in *n*+1 level layer and it ends. □

Theorem 3 (Property of Algorithm 1) :

For given maximum number of divergence V_k, the above algorithm is optimal in the meaning of minimization of a maximum length in $l_1, l_2, \cdots, l_D$. Moreover, both the maximum process length and the average process length are upper bounded by N that satisfies

$$\sum_{i=1}^{N} \log V_i \geq \log D \qquad (11)$$

□

If the sequence $V_1, V_2, V_3 \cdots$ is linear order and

$$V_1 = \alpha V, \qquad V_2 = \alpha^2 V, \qquad V_3 = \alpha^3 V, \cdots$$

are satisfied, then the maximum process length *K* is upper bounded by

$$K \leq \left\lceil \frac{-\log_V \alpha - 2 + \sqrt{8 \log_V \alpha \log_V D}}{2 \log_V \alpha} \right\rceil \qquad (12)$$

4.3 Formulation of Minimization of Average Process Length problem

In this section, let consider the problem to find the inventory tree minimizing the average process length

$$L_2 = \sum_{i=1}^{D} l_i p_i \qquad (13)$$

Again, we assume that the probability masses are ordered, so that

$$p_1 \leq p_2 \leq \cdots \leq p_D$$

Then, the optimal tree minimizing the objective function (13) satisfies the following theorem.

Theorem 4 (Property of Optimal Tree) :

Given the maximum number of divergence V_k with $V_1 \le V_2 \le V_3 \cdots$, then the optimal tree minimizing the average process length (13) satisfies the following properties :

1. If $p_i > p_j$, then $l_i \le l_j$
2. The levels of two nodes with equal probabilities each other are also same, or difference is at most 1 if different.
3. Defining K_U as a minimum k satisfying

$$\sum_{i=1}^{k} V_i - k + 1 \ge D \tag{14}$$

the maximum depth of the optimal tree K is upper bounded as $K \le K_U$

4. If the optimal tree is a complete tree, then the number of the leafs of the tree can be expressed by the form $V_1 + m_1(V_2 - 1) + m_2(V_3 - 1) + \cdots + m_{K_U}(V_{K_U} - 1)$ by using some natural numbers $m_1, m_2, \cdots, m_{K_U}$
5. Let T_{N-1} be an optimal tree with N-1 intermediate nodes for $p_1, p_2, \cdots, p_D$. The leaf level (depth) of p_1 is denoted by l. If a new tree T_N is made such that the minimum probability p_1 is divided to V_{l+1} leafs with probabilities p'_1, p'_2, $\cdots, p'_{V_{l+1}}$, where $\sum_i p'_i = p_1$, and add these V_{l+1} leafs into T_{N-1}, then T_N is also optimal tree for the probabilities set $\{p'_1, p'_2, \cdots, p'_{V_{l+1}}, p_2, p_3, \cdots, p_D\}$

□

Using the above properties, the optimal algorithm can be acquired in the following procedure.

Optimal Algorithm 2:

1. (Step1) For $p_1 \le p_2 \le \cdots \le p_D$, if $D \le V_1$, then correspond all leaf nodes to be at level 1 and end.
2. (Step2) Let minimum k satisfying the inequality (14) be K_U.
3. (Step3) For $i = 2,3,\cdots,K_U$, calculate

$$q_i = p_1 + p_2 + \cdots + p_{V_i}$$

Let C_i be a set union of a node set $Q_i = \{q_i, p_{V_i+1}, p_{V_i+2}, \cdots, p_D\}$ and sub-tree having $p_1, p_2, \cdots, p_{V_i}$ as a children of the same intermediate node. Keep a set of sets $R = \{C_i\}$.

4. (Step4) Sort the elements of Q_i in R as the small order. Let the number of q_i in ascending order be t_i. Let these numbering be $p_1 \le p_2 \le \cdots$ and replace them to Q_i.
5. (Step5) For all $j = 2,3,\cdots,K_U$ satisfying $V_j < t_i$ in all element sets $R = \{C_i\}$, calculate

$$q^i_j = p_1 + p_2 + \cdots + p_{V_j}$$

Let Q^i_j be $Q^i_j = \{q^i_j, p_{V_j+1}, p_{V_j+2}, \cdots, p_D\}$ and add Q^i_j into R.

6. (Step6) For all element sets $R = \{C_i\}$, if i satisfying $t_i \le V_{i-1}$ exists, then calculate

$$q^i = p_1 + p_2 + \cdots + p_{V_{i-1}}$$

and let Q^i be $Q^i = \{q^i, p_{V_{i-1}+1}, p_{V_{i-1}+2}, \cdots, p_D\}$. Remove C_i from R and add the set union of Q^i and sub tree representing q^i

7. (Step7) Repeat the calculations from Step4 to Step6 until a set of all leafs is empty.
8. (Step8) Select a tree with the smallest average length and it end. □

Theorem 5 (Property of Algorithm 2) :

For the given maximum number of divergence V_k, the above algorithm 2 is the optimal in a sense that minimization of an average process length L_2 is performed. □

4.4 Consideration

In the previous sub-section, the general Craft's inequality was shown and the optimal algorithms were derived to acquire the optimal tree structure in the meaning of both minimization of a maximum length in $l_1, l_2, \cdots, l_D$ and an average process length.

These results have a closed relation to the theory of encoding in the Information Theory. From the viewpoint of the generality of theory, the results in this paper are more general and include some results of source coding theory. In Section 4.3, the inventory tree minimizing the average process length was discussed. In the conventional source coding theory in Information Theory, the mean code length was an objective function. The formulation in this paper is an analogy of the source coding problem. The algorithm 2 is a generalized version of the Haffman code [7] in source coding.

From the viewpoint of logistics or supply chain management, the setting in this paper is idealized a little. However, the idealization brought us the new aspects of elegant and simple relation between multi-layered inventory or logistics problems and Information Theory. It may be necessary to loosen the condition and take up the problem by a setting similar with a real problem in a future study.

5 OPTIMIZATION OF GENERALITY OF INVENTORY WITH SOME RESTRICTION

5.1 Practical Aspects in the Multi-Layered Inventory Process Model

In Section 3, the generality of inventory was discussed. In practice, one of the most important criteria is Lead Time. Demand in the metropolitan area is more than that of the provinces usually. Therefore, the distribution center and the warehouse near the metropolitan area must shorten the Lead Time to demand in there. Thus, the distance should be considered and some structures in the inventory tree model cannot be taken in practical case.

On the other hand, only the fixed average demand was considered in the previous sections. However, it is actually necessary to correspond to the stationary change of demand. Generally speaking, the demand fluctuation can be absorbed to manage the demand for two or more shops together [1]. To do this, the distribution center and warehouse are significant in practice.

5.2 Definition of the Divergence Level of Inventory

In this section, let introduce a new entropy criterion representing effectiveness of management of many shop demand together.

Definition 1: For all $k \le K$, *the divergence level of inventory* at a node s_j^k is defined by the entropy

$$H\left(X_{k+1} \middle| s_j^k\right) = -\sum_i \left(\frac{p_i^{k+1}}{p_j^k}\right) \log \left(\frac{p_i^{k+1}}{p_j^k}\right) \qquad (15)$$

given s_j^k. *The divergence level* of *k*-th level inventories is defined by the total entropy

$$\sum_{s_j^k \in S_k} H\left(X_{k+1} \middle| s_j^k\right) \qquad (16)$$

□

5.3 Formulation

The divergence level should be large to manage a lot of demand points (shops) together and to absorb the demand fluctuation. This is because the children nodes must be divided as each node has a large probability mass. Therefore, the objective function

$$L_3 = \sum_{k=1}^{K-1} \sum_{s_j^k \in S_k} H\left(X_{k+1} \middle| s_j^k\right) \qquad (17)$$

can be introduced and an optimal problem is formulated. Here, the following result is derived.

Theorem 6 :

The minimization of an average process length L_2 is equivalent to the maximization of the total divergence level criterion L_3. □

The optimal algorithm 2 by the formulation in Section 4.3 can be applied to this optimization problem.

6 DISCUSSIONS

In this paper, the some idealized setting model is proposed. In practice, the criterion such as

$$L = \alpha L_1 + (1-\alpha) L_2 \qquad (18)$$

can be applied to find a practical solution.

The expression of tree structure as a probability model brings us some new aspects. In many conventional studies, the social models, economic models, and management models, i.e., the multi-layered inventory model, have been studied from economic point of view. In these settings, the concept of cost was important. Of course, these formulations are significant in practice. However, the formulation in this paper may be useful to grasp the essentials of new business model.

7 CONCLUSION

In this paper, the multi-layered inventory process model from the viewpoint of Information Theory was discussed. The problems of the multi-layer inventory process have been explained as analogies with entropy and source coding problems in Information Theory. As the results, some interesting aspects were found in both the multi-layered inventory model and Information Theory.

The Inventory control is essentially control problem. In fact, many studies based on control theory can be found. It may be possible to establish the general theory including control theory and information theory in the future study.

REFERENCES

[1] David Simchi-Levi, Philip Kaminsky, Edith Simchi-Levi: *Designing and Managing the Supply Chain*, The McGraw-Hill Companies, Inc. (2000)

[2] M.Sharafali, H.C.Co: "Some Models for Understanding the Cooperation between the Supplier and the Buyer", *International Journal of Production Reserch*, Vol.38, No.15, pp.3425-3449, (2000)

[3] K.L.Cheung, A.X.Zhang: "The Impact of Inventory Information Distortion Due to Customer Order Cancellations", *Naval Res. Logistics*, Vol.46, No.2, pp.213-131, (1999)

[4] H.Masuda, M.Matsui, H.Yamashita: "The Optimal Control Problem of Associated Make-To-Order Supply Chain Management", *The 1st International Congress on Logistics and SCM Systems* (*ICLS-2004*), pp.40-47, (2004)

[5] N.Kawai, T.Masui, M.Goto: "The Model of analysys in the information chain", *The 1st International Congress on Logistics and SCM Systems*(*ICLS-2004*), pp.161-166, (2004)

[6] "In 1998, A report of basic plan to develop the business model in physical distribution (ALIS)", *Japan Institute of Logistics System*, published by 1999/3,

[7] C.E.Shannon: "A Mathematical Theory of Communication", *Bell Sys. Tech. J.* 27, pp.379-423, pp.623-656, (1948)

[8] Tomas M. Cover, Joy A. Thomas: *Elements of Information Theory*, John Wiley & Sons, INC, (1991)

[9] Imre Csiszar, Janos Korner: *Information Theory – Coding Theorems for Discrete Memoryless Systems*, Akademiai Kiado, Budapest, Hungary, (1986)

[10] Richard E. Blahut: *Principles and Practice of Information Theory*, Addison-Wesley Publishing Company, New York, (1991)

[11] D. Hankerson, G.A.Harris, P.D.Johnson,Jr. : *Introduction to Information Theory and Data Compression*, CRC Press, New York, (1998)