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On Correctable Burst-Erasure Lengths for LDPC Codes with Column Permuted Parity-Check Matrices

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Abstract— We derive the correctable burst-erasure lengths for low-density parity-checkparity-check (LDPC) codes. The columns of the parity-check matrix of LDPC codes are permuted to increase the distance between adjacent elements (DBEs) which are defined as a number of symbol positions between elements 1 at each row of the parity-check matrix. The column permutation method can change the burst-erasure correction capabilities by both the sum-product decoding algorithm and maximum-likelihood decoding algorithm without changing the random error correction capability. We compare the correctable burst-erasure lengths with the performance of computer-generated codes.

Keywords— low-density parity-check code, burst erausre channel, sum-product decoding

1 Introduction

The combination of low-density parity-check (LDPC) codes with the sum-product (SP) decoding algorithm achieves high performance with low decoding complexity [1]. Most of studies of LDPC codes assume random errors or random erasures. When we consider practical applications of LDPC codes, we must take into account correction capabilities of not only random errors or erasures but also burst ones. Present authors have shown that for two or more solid burst-erasures[1], the codes constructed by a column permutation based on increasing distance between elements (DBEs), which are a number of symbol positions between adjacent elements 1 at each row of the parity-check matrix, have good performance of erasure correction [8], while there is no degradation in the performance over random erasure channels.

In this paper, we derive the correctable burst-erasure lengths for LDPC codes. The columns of the parity-check matrix of LDPC codes are permuted to increase the DBEs, which can easily change the performance of the codes both by the SP decoding algorithm and maximum likelihood decoding (MLD) algorithm. We compare the correctable burst-erasure lengths with the actual performance of computer-generated codes.

This paper is organized as follows. In Section 2, we describe LDPC codes and decoding for the erasure channel. In Section 3, we describe a column permutation method and derive the correctable burst-erasure lengths. In Section 4, we compare the correctable burst-erasure lengths obtained in Section 3 with the performance of computer-generated codes. Finally, conclusion is given in Section 5.

2 Preliminaries

2.1 LDPC Codes

Let $H = [H_{mn}]$, $m \in [1, M]$, $n \in [1, N]$, be a parity-check matrix whose row and column lengths are M and N, respectively[2]. Let w_r and w_c be the row and the column weight of H, respectively. Let M be the number of rows of H which is given by $M = Nw_c/w_r$. In this paper, we consider binary regular (N, w_r, w_c) LDPC codes to simplify the discussion.

A parity-check matrix is represented by the *bipartite graph*, which consists of two types of nodes called *check nodes* indexed by position of rows, and *symbol nodes* indexed by position of columns. A check node m and a symbol node n are connected with an edge if and only if $H_{mn} = 1$. A loop in the bipartite graph is a closed pass that starts from a symbol node and returns to the same symbol node through edges without passing the same edges more than once. A length of a loop is a number of edges of the closed pass. Let g be a girth of H which is the smallest length of loops for H. We define a span of the loop as the difference of symbol positions between leftmost element 1 and rightmost element 1.

We assume a codeword $c = (c_1, c_2, \dots, c_N) \in \{0, 1\}^N$ of a LDPC code of length N is transmitted through an erasure channel. c is disturbed by a sequence from the channel $e = (e_1, e_2, \dots, e_N) \in \{0, \epsilon\}^N$ where ϵ denotes an erasure, and the decoder receives a sequence $y = c + e$. The addition of a binary symbol and the erasure symbol is defined as $0 + \epsilon = \epsilon$, $1 + \epsilon = \epsilon$. The decoder estimates the transmitted codeword from the received sequence.

Let $\mathcal{N} = \{1, 2, \dots, N\} = [1, N]$ be the index set of symbol positions. And let $\mathcal{E} \subseteq \mathcal{N}$ and $\bar{\mathcal{E}} = \mathcal{N} \setminus \mathcal{E}$ be the index sets of erased symbol positions and known symbol positions, respectively. From the definition of the parity-check matrix, we can write

$$cH^{\mathrm{T}} = c_{\mathcal{E}} H_{\mathcal{E}}^{\mathrm{T}} \oplus c_{\bar{\mathcal{E}}} H_{\bar{\mathcal{E}}}^{\mathrm{T}} = 0, \tag{1}$$

where $c_{\mathcal{E}}$ and $H_{\mathcal{E}}$ are a subvector of c and a submatrix of c and H which consist of those columns indexed by

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[1] A "solid burst-erasure" of length T stands for consecutive T erasures.

[2] For two integers i and j ($i \le j$), $[i, j]$ denotes the set of integers from i to j.

$\mathcal{E}$, respectively. Since $c_{\bar{\mathcal{E}}} H_{\bar{\mathcal{E}}}^T$ is known to a decoder,

$$c_{\mathcal{E}} H_{\mathcal{E}}^T = c_{\bar{\mathcal{E}}} H_{\bar{\mathcal{E}}}^T = s', \quad (2)$$

where $s' = (s'_1, s'_2, \dots, s'_M) \in \{0,1\}^M$ is a syndrome sequence calculated by $c_{\bar{\mathcal{E}}} H_{\bar{\mathcal{E}}}^T$. Maximum likelihood decoding (MLD) algorithm obtains the erased (unknown) sequence $c_{\mathcal{E}}$ from the simultaneous equations $c_{\mathcal{E}} H_{\mathcal{E}}^T = s'$. Therefore MLD can decode the received sequence correctly iff $rank[H_{\mathcal{E}}] = |\mathcal{E}|$ where $rank[A]$ denotes the rank of a matrix A. Note that the rank of $H_{\mathcal{E}}$ is given by an erasure pattern $\mathcal{E}$, so this is not equal to the rank of parity-check matrix H.

The SP decoding algorithm on the binary erasure channel fails in decoding when a subset of erased symbols have a *stopping set.*

Definition 1. [Stopping set [2]] *Choose some columns of H to make a submatrix. A stopping set $S \in \mathcal{E}$ is a subset of symbol positions such that weights of all rows of this submatrix H_S are at least two. A union of stopping sets is also a stopping set, so any submatrix of a parity-check matrix has a unique maximal stopping set.* □

2.2 Reasonable Maximum Burst-Erasure Length

Definition 2. [$L_{\max}$ [4]] *$L_{\max}$ is a reasonable maximum burst-erasure length which can be decoded by the SP decoding algorithm when only one burst-erasure with length smaller than or equal to $L_{\max}$ occurred.* □

Note that $L_{\max}$ is a correctable capability for only one solid burst-erasure. M. Yang et al. have proposed the $L_{\max}$ algorithm which can evaluate a correctable burst-erasure length for a given parity-check matrix of LDPC codes by an exhaustive search. See [4] for details.

3 Correctable Burst-erasure Lengths and Column Permutation Method

3.1 Definitions of DBEs

We define the following set for $m \in [1, M]$.

$$\mathcal{A}(m) \triangleq \{n : H_{mn} = 1\} = \{n_{m,1}, n_{m,2}, \dots, n_{m,w_r}\},$$

where $n_{m,1} < n_{m,2} < \cdots < n_{m,w_r}$.

We define the *distance between elements* (DBE) as the number of symbol positions between adjacent elements 1 at each row of the parity-check matrix.

Definition 3. [DBE] *The DBEs $d_{m\gamma}$, $m \in [1, M]$, $\gamma \in [1, w_r - 1]$, the minimum value of DBEs $D_{\min}$, and an arithmetic average value of DBEs D_{ave} are defined by the following equations, respectively:*

$$d_{m\gamma} \triangleq n_{m,\gamma+1} - n_{m,\gamma}, \quad (3)$$

$$D_{\min} \triangleq \min_{m,\gamma}\{d_{m\gamma}\}, \quad (4)$$

$$D_{\text{ave}} \triangleq \frac{1}{M(w_r-1)} \sum_{m=1}^{M} \sum_{\gamma=1}^{w_r-1} d_{m\gamma}. \quad (5)$$

3.2 Column Permuted Parity-Check Matrix [8]

For a parity-check matrix, the DBEs are changed by column permutation. We can state structures of the parity-check matrix that maximizes the value of D_{ave}. To increase DBEs, we consider the following two conditions: (i) D_{ave} has a large value, (ii) $D_{\min} > \delta$ where δ is some positive constant. From Eq. (3), we have

$$\sum_{\gamma=1}^{w_r-1} d_{m\gamma} = n_{m,w_r} - n_{m,1}. \quad (6)$$

We can easily see that D_{ave} depends on a difference of column positions between the leftmost element 1 and the rightmost element 1 at each row of a parity-check matrix since we have

$$D_{\text{ave}} = \frac{1}{M(w_r-1)} \sum_{m=1}^{M} (n_{m,w_r} - n_{m,1}). \quad (7)$$

Therefore the condition (i) implies that the sum of DBEs ($n_{m,w_r} - n_{m,1}$ for $\forall m$) has a large value.

We now let the leftmost (rightmost) symbol position at each row of H be as small (large) as possible. Let $r' \triangleq M \bmod w_c$ and $\rho' \triangleq \frac{M-r'}{w_c} = \frac{N}{w_r} - \frac{r'}{w_c}$. Assume that H has a following form:

$$H \triangleq [H_{\text{lef}}, H_{\text{mid}}, H_{\text{rig}}]. \quad (8)$$

H_{lef} and H_{rig} are $M \times \rho'$ matrices such that the weights of r' rows and $M - r'$ rows of these matrices are 0 and 1, respectively, and the weights of columns of those are w_c. H_{lef} constitutes the leftmost ρ' columns of H. Assume that column positions of element 1 at those columns are $n_{m,1}$, the leftmost element 1 at each row. Similarly, H_{rig} constitutes the rightmost ρ' columns of H. Assume again that column positions of element 1 at those columns are n_{m,w_r}, the rightmost element 1 at each row. Then the following lemma holds.

Lemma 1. *The parity-check matrix H that has the form of Eq.* (8) *maximizes the right-hand side of Eq.* (7). □

To modify the parity-check matrix of LDPC codes suitable for burst-erasures, we permute the columns of a parity-check matrix of LDPC codes to have the form Eq. (8) as nearly same as possible. Detailed description of the column permutation algorithm is omitted. (See [8] for details.)

3.3 Correctable Burst-Erasure Length

In this section, we derive the correctable burst-erasure lengths of LDPC codes with column permuted parity-check matrices by using the SP decoding algorithm. Since the columns of the parity-check matrix of the LDPC codes is permuted by the permutation method to have large values of DBEs [8], the $D_{\min}$ of column permuted parity-check matrix is a large value. We assume that single solid burst-erasure of length B has been occurred at the received sequence.

Lemma 2. [5] *The submatrix of H, whose column positions are indexed by a stopping set, has at least one loop when $w_c \geq 2$.* □

Since the span of a loop is equal to or greater than $D_{\min}$, it is guaranteed that any $D_{\min}$ consecutive column positions of submatrices of H are free from any loops. We first show a correctable burst-erasure length of the LDPC codes with the girth 4.

Lemma 3. *(N, w_r, w_c) LDPC codes with $w_c \geq 2$ and the girth $g = 4$ can correct single solid burst-erasure of length B, which is given by the following equation:*

$$B \leq D_{\min}. \tag{9}$$

Proof. If two column vectors of H are the same, then these two columns constitute a loop of length 4. Since the difference of these two column positions is equal to or greater than $D_{\min}$, Eq. (9) holds. □

The number of loops of the length 4 is relatively small when using LDPC codes with large code length. Although the girth of H is 4, it has been probably occurred that the girth of $H_{\mathcal{E}}$ is greater than 4. Therefore the correctable burst-erasure length, which is obtained in Lemma 3, can be seen as the worst case.

We next show that the correctable burst-erasure length in the case of $g > 4$ is nearly twice as large as that in the case of $g = 4$. In this paper, we show in the case of (N, w_r, w_c) LDPC codes with $w_c = 3$.

Theorem 1. *(N, w_r, w_c) LDPC codes with $w_c = 3$ and girth $g \geq 6$ can correct single solid burst-erasure of a length B which satisfies the following equations:*

$$B \leq \min\left\{2D_{\min}+2, \left(\left\lceil \frac{D_{\min}-1}{2} \right\rceil + 1\right) \times 4 - 1\right\}. \tag{10}$$

Proof. From Lemma 2, $H_{\mathcal{S}}$ has at least one loop. We consider the cases that lengths of loops are 6 and 8[3].

We first assume a loop of the length 6 whose span takes the smallest value and this loop of the length 6 is in symbol positions at n_1, n_2, and n_3, and in row positions at m_1, m_2, and m_3 where $n_2 - n_1 = D_{\min}$ and $n_3 - n_2 = D_{\min}$. For example, $H_{m_1 n_1} = 1$, $H_{m_1 n_2} = 1$, $H_{m_2 n_2} = 1$, $H_{m_2 n_3} = 1$, $H_{m_3 n_1} = 1$, and $H_{m_3 n_3} = 1$.

[3] The case when the length of a loop is greater than 8, a correctable burst-erasure length is always bigger than right hand side of Eq. (10).

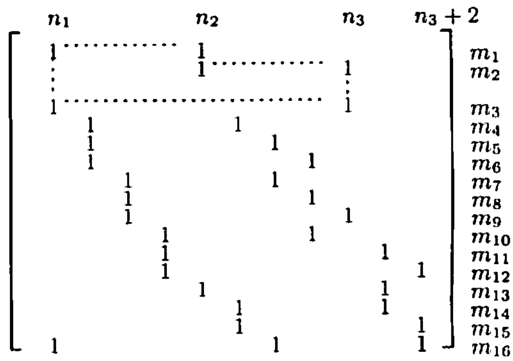

Figure 1: An example of a parity-check matrix when $D_{\min} = 3$. n_i, $i = 1, 2, 3$, and m_j, $j = 1, 2, \ldots, 16$, denote the symbol positions and row positions, respectively. The set of row positions $\mathcal{M} = \{m_4, m_5, \ldots, m_{12}\}$.

The span of this loop is $2D_{\min}$. We show that a set of symbol positions at $[n_1, n_3]$ is not a stopping set, and show that sets of symbol positions at $[n_1+i-2, n_3+i]$, $i = 0, 1, 2$, can be a stopping set. We consider the case of $i = 2$ such that the set of symbol positions at $[n_1, n_3 + 2]$. Assume that the element 1s at the symbol positions $[n_1 + 1, n_2 - 1]$ are all in the different positions of rows. We denotes the set of positions of these rows as $\mathcal{M}$ and $|\mathcal{M}| = w_c \times (D_{\min} - 1)$. We next let the adjacent elements 1 at $\mathcal{M}$ with keeping $D_{\min}$ as the same value and without making any loops of the length 4. We can let one element 1 at symbol position $n_2 + 1$, since adjacent element 1 of this element 1 can be at a symbol position $n_1 + 1$. In the same way, we can let two elements 1 at symbol position $n_2 + 2$. We let three elements 1 at each symbol position in $[n_2 + 3, n_3 - 1]$ and let element 1 at symbol position n_3. Until now, we make $w_c \times (D_{\min} - 2) + 1$ rows as weight two. We enforce to set remaining two elements 1 at symbol positions $n_3 + 1$ and $n_3 + 2$ and at row positions in $\mathcal{M}$ with their weights are 1. This makes clear that the set of symbol positions $[n_1, n_3 + 1]$ cannot be a stopping set. Finally we can set remaining elements 1 with keeping the weights of all rows greater than 2 at symbol positions $[n_1, n_3 + 2]$, we can make a stopping set. The span of this stopping set is $2D_{\min} + 2$, and for any subset of this $2D_{\min} + 2$ consecutive symbol positions, there are no stopping sets. Fig. 1 shows an example when $D_{\min} = 3$. Then we can correct a solid burst-erasure of the length B such that:

$$B \leq 2D_{\min} + 2. \tag{11}$$

By a similar same argument when the length of the loop is 8, we obtain:

$$B \leq \left(\left\lceil \frac{D_{\min}-1}{2} \right\rceil + 1\right) \times 4 - 1. \tag{12}$$

Taking the minimum values of Eqs. (11) and (12), a

Table 1: The values of D_{ave} ($D_{\min}$) for each code $\mathcal{C}_4$

	Original	DBE	$L_{\max}$	Small
$\mathcal{C}_4^1$	62.4 (1)	82.4 (53)	63.5 (1)	54.1 (1)
$\mathcal{C}_4^2$	62.8 (1)	82.4 (53)	65.1 (1)	54.0 (1)
$\mathcal{C}_4^3$	61.2 (1)	82.3 (53)	62.5 (1)	53.5 (1)

Table 2: The values of $L_{\max}$ for each code $\mathcal{C}_4$

	Original	DBE	$L_{\max}$	Small
$\mathcal{C}_4^1$	163	190	201	137
$\mathcal{C}_4^2$	157	192	201	137
$\mathcal{C}_4^3$	147	198	201	129

Table 3: The values of D_{ave} ($D_{\min}$) for each code $\mathcal{C}_6$

	Original	DBE	$L_{\max}$	Small
$\mathcal{C}_6^1$	62.0 (1)	82.5 (53)	64.5 (1)	53.7 (1)
$\mathcal{C}_6^2$	64.4 (1)	82.4 (53)	66.4 (1)	55.9 (1)
$\mathcal{C}_6^3$	64.9 (1)	82.3 (53)	66.4 (1)	56.7 (1)

Table 4: The values of $L_{\max}$ for each code $\mathcal{C}_6$

	Original	DBE	$L_{\max}$	Small
$\mathcal{C}_6^1$	173	191	208	131
$\mathcal{C}_6^2$	165	193	207	137
$\mathcal{C}_6^3$	164	189	209	125

correctable burst-erasure length satisfies

$$B \leq \min\left\{2D_{\min}+2, \left(\left\lceil\frac{D_{\min}-1}{2}\right\rceil+1\right)\times 4-1\right\}. \quad (13)$$

We can obtain Eq. (10). □

The case of $w_c > 3$ is remained for a further investigation.

4 Comparison with Some LDPC Codes

In this section, we compare the correctable burst-erasure lengths of the LDPC codes with that of computer generated LDPC codes.

4.1 Code Parameters

We use LDPC codes of $N = 500$, $w_r = 6$, and $w_c = 3$ with $g = 4$ (denoted by "$\mathcal{C}_4$") and $g = 6$ (denoted by "$\mathcal{C}_6$"). We construct three codes (denoted by "$\mathcal{C}_g^1$", "$\mathcal{C}_g^2$", and "$\mathcal{C}_g^3$") by different seeds of random generator (these original codes are denoted by "Original"). Those "Original" codes are constructed carefully that the values of $L_{\max}$ of these codes are relatively large. For each code, we permute columns of a parity-check matrix of the original codes. The permutation methods are 1) based on DBEs [8] (denoted by "DBE"), 2) based on increasing $L_{\max}$ [6] (denoted by "$L_{\max}$")[4], and 3) to have values of many DBEs 1 (denoted by "Small").

4.2 Values of DBEs and Comparision

We show the values $L_{\max}$, D_{ave}, and $D_{\min}$ for $\mathcal{C}_4$ and $\mathcal{C}_6$ in Tables 1 ~ 4, respectively. From these tables, although the girth of $\mathcal{C}_4$ and $\mathcal{C}_6$ are different, the values $L_{\max}$, D_{ave}, and $D_{\min}$ are almost the same. Since the values of $D_{\min}$ of the codes "DBE" of $\mathcal{C}_6$ are all 53, the correctable burst-erasure length 107 is obtained from Theorem 1. The values of $L_{\max}$ of $\mathcal{C}_6^1 \sim \mathcal{C}_6^3$ are 191, 193, and 189, respectively. The gap between the correctable burst-erasure length, which is derived in Theorem 1, and computer-generated codes exist with not so tight, since the bounds which we derived are considered the worst case.

[4] Note that this column permutation method may delete some columns for increasing $L_{\max}$ [6]. However, we do not delete any columns for a fair comparison.

5 Conclusion

We derive the correctable burst-erasure lengths for low-density parity-check (LDPC) codes with column permuted parity-check matrices when the column weight is 3. The result is not tight compared with the performance of the computer generated codes. Burst-erasure correction capability of the codes, which is obtained by the column permutation method [8], is especially good for two or more (multiple) solid burst-erasures. We did not derive the correction capabilities in such cases and it remains for further investigations.

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