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A Combined Matrix Ensemble of Low-Density Parity-Check Codes for Correcting a Solid Burst Erasure

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SUMMARY A new ensemble of low-density parity-check (LDPC) codes for correcting a solid burst erasure is proposed. This ensemble is an instance of a combined matrix ensemble obtained by concatenating some LDPC matrices. We derive a new bound on the critical minimum span ratio of stopping sets for the proposed code ensemble by modifying the bound for ordinary code ensemble. By calculating this bound, we show that the critical minimum span ratio of stopping sets for the proposed code ensemble is better than that of the conventional one with keeping the same critical exponent of stopping ratio for both ensemble. Furthermore from experimental results, we show that the average minimum span of stopping sets for a solid burst erasure of the proposed codes is larger than that of the conventional ones.

key words: *low-density parity-check code, solid burst erasure, stopping set, minimum span, belief-propagation decoding*

1. Introduction

A combination of low-density parity-check (LDPC) codes and the belief-propagation (BP) decoding algorithm attains a good decoding performance, and its decoding complexity is quite small [1], [2]. Most of studies of LDPC codes assume random errors or random erasures. When we consider using LDPC codes in a practical situation, such as the packet based transmission used for the wireless communications, we must take into account correction capabilities of not only random errors or erasures but also burst ones. For the packet based transmission used for the wireless transmission, the burst erasure has been occurred under bad transmission environment [9]. In this paper, we only focus on the solid burst erasure* assuming the worst case for the burst erasure.

Using LDPC codes for correcting single or multiple solid burst erasures has been studied in [6], [10], [11]. M. Yang et al. have proposed the $L_{\max}$ algorithm, which can evaluate a maximum correctable length of solid burst erasure for a given parity-check matrix of LDPC codes by an exhaustive search method. T. Wadayama [6] and E. Paolini and M. Chiani [10] have proposed column permutation algorithms which can increase the value of $L_{\max}$ for a given parity-check matrix of an LDPC code. The present authors also have proposed a column permutation algorithm which can improve the performance for correcting multiple solid burst erasures [11].

On the other hand, for analyzing the performance of LDPC codes, the approach of the code ensemble analysis has been taken [1], [2], [5], [7]–[9], [12]. The analysis of the LDPC code ensemble for the BP decoding algorithm over the erasure channel has been well studied [3], [5], [7], [8], [12]. Wadayama has formulated a method for calculating the bound of the minimum length of uncorrectable single solid burst erasure for the LDPC code ensemble by the BP decoding algorithm [9].

In this paper, a new ensemble of LDPC codes suitable for correcting a solid burst erasure is proposed. This ensemble is an instance of a combined matrix ensemble [12] obtained by concatenating the plural number of LDPC matrices and is a subclass of LDPC codes defined on the regular Tanner graph. We derive a new bound on the critical minimum span ratio of stopping sets for the proposed code ensemble by modifying the derivation method in [9]. By calculating the above bound, we show that it is better than the standard LDPC codes keeping the critical exponent of stopping ratio for both ensembles equal. Furthermore from experimental results, we show that the average minimum span of stopping sets of a solid burst erasure for both the proposed code and the code in [11] have larger than that for the conventional one.

This paper is organized as follows. In Sect. 2, we describe the LDPC codes, the Tanner graphs, the LDPC code ensemble, the stopping set, and the span of stopping sets. In Sect. 3, we propose the new ensemble of LDPC codes. We explain overview of the structure of the proposed codes in Sect. 3.1. We mention the concept in Sect. 3.2. We formulate the average stopping set distribution for the proposed code ensemble in Sect. 3.3. We formulate a bound on the average minimum span of stopping set for the proposed code ensemble in Sect. 4.1 and then derive an asymptotic property of this bound in Sect. 4.2. In Sect. 5, we describe a relationship between the proposed codes and the codes obtained by a column permutation algorithm discussed in [11]. Finally, some example of numerically calculated and experimental results are presented in Sect. 6 and the conclusion is given in Sect. 7.

2. Preliminaries

2.1 LDPC Code and Tanner Graph

Let $H = [H_{mn}]$, $m \in [1, M]$, $n \in [1, N]$, be a parity-check

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*"solid" burst erasure means that the consecutive bit positions are all in erasures.

matrix of an LDPC code, where M and N denote the length of row and column, respectively[†]. We call the number of element ones in each row (column) of H the weight of a row (column). The designed rate R' of the LDPC code is given by $R' = 1 - \frac{M}{N}$.

It is convenient to specify an LDPC codes by representing the Tanner graph G. Let $G = (V \cup C, E)$ be the Tanner graph where $V = \{v_1, v_2, \ldots, v_N\}$ and $C = \{c_1, c_2, \ldots, c_M\}$, $V \cap C = \phi$, are the sets of variable nodes and check nodes, respectively, and E denotes a set of edges. A variable node $v_n \in V$ corresponds to a column at position n in H and a check node $c_m \in C$ corresponds to a row at position m in H. The equality $H_{mn} = 1$ holds if the nodes v_n and c_m are adjacent each other for an odd number edges, and the equality $H_{mn} = 0$ holds otherwise. We call the number of edges from a variable node (check node) the degree of variable node (check node). The (c, d) regular Tanner graph is defined by a graph such that the degree of each variable node is constant c and the degree of each check node is d [5]. The total number of edges in G is given by $|E| = Nc = Md$. In this paper, we deal with the LDPC codes corresponding to the regular Tanner graph as studied in [9]. Hereafter, we refer to the regular Tanner graph as the Tanner graph for simplicity.

2.2 Standard Tanner Graph Ensemble

For each variable (check) node, we assign c variable (d check) sockets. The total number of variable (check) sockets is therefore $|E|$. An ensemble[††] of the Tanner graph is obtained by choosing a permutation function $\pi : \{1, 2, \ldots, |E|\} \to \{1, 2, \ldots, |E|\}$ with assigning uniform probability from the space of all permutations of $\{1, 2, \ldots, |E|\} \to \{1, 2, \ldots, |E|\}$. For each $i = 1, 2, \ldots, |E|$, the variable node associated with the ith variable socket to the check node associated with the $\pi(i)$th check socket are adjacent to each other with the edge. Note that there may be multiple edges which are incident to the same pair of a variable node and a check node [5]. Let $\mathcal{G}(N, c, d)$ be all the collections of the (c, d) Tanner graph G with N variable nodes.

Since a graph G which belongs to $\mathcal{G}(N, c, d)$ is assigned with equal probability, the equality $\Pr(G) = 1/|\mathcal{G}(N, c, d)|$ holds. In this paper, we call the Tanner graph $G \in \mathcal{G}(N, c, d)$, the standard Tanner graph.

Figure 1 shows a standard Tanner graph ensemble. The solid circles and solid squares with the white background indicate the variable nodes and check nodes, respectively. The squares with black background indicate the variable sockets or the check sockets. An example of multiple edges is given in Fig. 2.

It can be easily verified that $\mathcal{G}(N, c, d)$ has the following property.

Lemma 1: [8] For $\mathcal{G}(N, c, d)$, we have $|\mathcal{G}(N, c, d)| = (Nc)!$. □

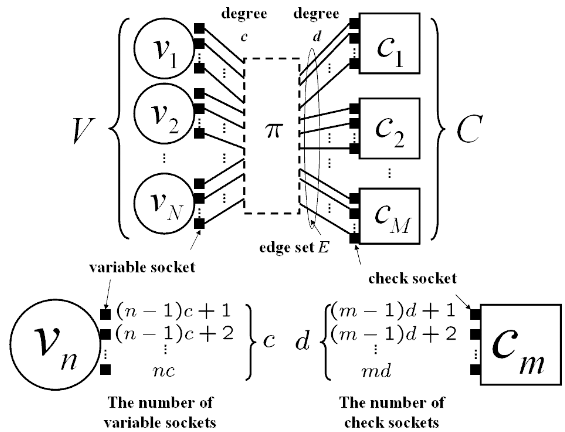

Fig. 1 Standard Tanner graph ensemble.

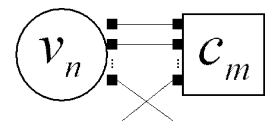

Fig. 2 An example of the multiple edges.

2.3 Stopping Set and Span of Stopping Sets

For a subset of the bit positions $\mathcal{X} \subseteq [1, N]$, let $V_{\mathcal{X}}$ denote a subset of the variable nodes V whose positions are indexed by $\mathcal{X}$. For a given $\mathcal{X}$, let $E_{\mathcal{X}}$ denote a subset of the edge set E which is connected to $V_{\mathcal{X}}$. We define a stopping set which is an important measure for the erasure correction by using LDPC codes.

Definition 1 (Stopping set [3]): For a subgraph $G_S = (V_S \cup T(V_S), E_S)$ of a graph G where S and $T(V_S)$ denote a subset of the bit positions and a subset of neighboring check nodes for V_S in G, respectively. We call S a stopping set if all the check nodes in $T(V_S)$ connect to V_S with at least two edges. □

Hereafter we consider the non-empty stopping set. Let $\mathcal{E}$ denote a set of erased bit positions of a channel output (received) sequence. If $\mathcal{E}$ contains some non-empty stopping set $S \subseteq \mathcal{E}$, then the received sequence with the erased bits in $\mathcal{E}$ cannot be corrected by the BP decoding algorithm. Let $\mathcal{S}(G)$ be all the collections of non-empty stopping sets for a graph G. For some stopping set $S \in \mathcal{S}(G)$, we call the value $|S|$ stopping set weight.

Definition 2 (Consecutive bit positions): For $L \in [1, M + 1]$ and $n \in [L, N]$, let $S_{n,L} = \{n - L + 1, n - L + 2, \ldots, n\}$ denote a set of consecutive bit positions of length L with its rightmost position n. □

Using Definition 2, we define a solid burst erasure.

[†]For two integers i and j ($i \leq j$), $[i, j]$ denotes the set of integers from i to j.

[††]More precisely, an ensemble of the Tanner graph is the set of pairs $(G, \Pr(G))$ where $\Pr(G)$ expresses the probability of G.

Definition 3 (Solid burst erasure): For $L \in [1, M+1]$ and $n \in [L, N]$, a solid burst erasure of length L at the set of positions $S_{n,L}$ is a channel output sequence with the erased bits only at $S_{n,L}$ such as $\mathcal{E} = S_{n,L}$. □

From Definitions 1 and 3, the burst erasure spanned by $S_{n,L}$ cannot be corrected by the BP decoding algorithm if a subset of $S_{n,L}$ is some non-empty stopping set in $\mathcal{S}(G)$ of a graph G. By using this fact, we define the burst erasure correction measure as follows:

Definition 4 (Span of stopping sets [9]): For a given graph G, the minimum span of stopping sets $\mu(G)$ is defined as

$$\mu(G) = \min_{\substack{L\in[1,M+1],\, n\in[L,N] \\ S_{n,L}\supseteq X,\, X\in\mathcal{S}(G)}} |S_{n,L}|. \tag{1}$$

□

In other words, $\mu(G)$ is the minimum length of uncorrectable solid burst erasures for a given Tanner graph $G \in \mathcal{G}(N, c, d)$ of the LDPC codes by executing the BP decoding algorithm. In [4], Yang et al. have proposed the $L_{\max}$ algorithm, which can evaluate the maximum value of correctable length of a solid burst erasure (the value $\mu(G) - 1$) for a given Tanner graph G by executing the BP decoding algorithm.

2.4 Notations

For a, $0 \le a \le 1$, let $H(a)$ be the binary entropy function defined by $H(a) = -a \log a - (1-a)\log(1-a)$. Note that we use the natural logarithm in this paper. For a_i such that $0 \le a_i \le 1$ with $i = 1, 2, 3$ and $\sum_{i=1}^{3} a_i = 1$, let $H(a_1, a_2, a_3) = -\sum_{i=1}^{3} a_i \log a_i$ be the ternary entropy function. The binomial coefficient and multinomial coefficient are denoted by

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad \binom{n}{k_0, k_1, k_2} = \frac{n!}{k_0!k_1!k_2!},$$

where k_0, k_1, and k_2 satisfy $k_0 + k_1 + k_2 = n$. Let $p(x)$ be a polynomial with non-negative coefficients such that $p(x) = \sum_{i=\mathrm{Ldeg}\, p(x)}^{\mathrm{Mdeg}\, p(x)} p_i x^i$ where p_i denotes the coefficient of x^i, and $\mathrm{Ldeg}\, p(x)$ and $\mathrm{Mdeg}\, p(x)$ denote the smallest and the largest index i, respectively, for which the coefficient is nonzero. For a polynomial $p(x)$, let $\mathrm{coef}[p(x), x^i]$ denote the coefficient of x^i. For a condition P, the function I[P] represents the indicator function, which takes 1 if the P is true and takes 0 otherwise.

For an asymptotic analysis, the following lemma will be used.

Lemma 2 ([5]): For some rational number $\theta_1 > 0$, assume $p(x)$ such that polynomial $p(x)^{\theta_1}$ has nonnegative coefficients. For some rational number $\theta_2 > 0$, let $N_i = \theta_2 N$ be series of all index i such that $\theta_2 N$ is a natural number. If $\mathrm{Ldeg}\, p(x)^{\theta_2} < \theta_1\theta_2 < \mathrm{Mdeg}\, p(x)^{\theta_2}$, then

$$\lim_{N_i\to\infty} \frac{1}{N_i} \log \mathrm{coef}[p(x)^{N_i}, x^{\theta_2 N_i}] = \log \frac{p(x_0)}{x_0}, \tag{2}$$

where x_0 is the only positive solution to $\frac{xp'(x)}{p(x)} = \theta_1$. $p'(x)$ is obtained by differentiating $p(x)$ with respect to x. □

2.5 Upper Bound on the Average Minimum Span of Stopping Sets

An upper bound on the average minimum span of stopping sets for the standard Tanner graph ensemble has been derived in [9] in which a method for analysing the average stopping set distribution† is used.

For a given graph G, let $N_{S_{L,L}}(G)$ be the number of non-empty stopping set $S \in \mathcal{S}(G)$ included in the set of consecutive bit positions $S_{L,L}$. Similarly for a given graph G and for $n \in [L+1, N]$, let $M_{S_{n,L}}(G)$ denotes the number of non-empty stopping set $S \in \mathcal{S}(G)$ included in the set of consecutive bit positions $S_{n,L}$ with a position n which is included in S.

Lemma 3 ([9]): $N_{S_{L,L}}(G)$ for $L \in [1, M+1]$ and $M_{S_{n,L}}(G)$ for $n \in [L+1, N]$ satisfy the following inequalities:

$$\begin{aligned}
\Pr[N_{S_{L,L}}(G) \ge 1] &= \sum_{G\in\mathcal{G}(N,c,d)} \Pr(G) I[N_{S_{L,L}}(G) \ge 1] \\
&\le \Bigl\{ \sum_{G\in\mathcal{G}(N,c,d)} \Pr(G) N_{S_{L,L}}(G) \Bigr\} / 1 \\
&= \sum_{w=1}^{L} \binom{L}{w} W(w), \qquad (3) \\
\Pr[M_{S_{n,L}}(G) \ge 1] &= \sum_{G\in\mathcal{G}(N,c,d)} \Pr(G) I[M_{S_{n,L}}(G) \ge 1] \\
&\le \Bigl\{ \sum_{G\in\mathcal{G}(N,c,d)} \Pr(G) M_{S_{n,L}}(G) \Bigr\} / 1 \\
&= \sum_{w=1}^{L} \binom{L-1}{w-1} W(w), \qquad (4)
\end{aligned}$$

where $W(w)$ is given by

$$W(w) = \mathrm{coef}\Bigl[((1+x)^d - dx)^M, x^{wc} \Bigr] \times \binom{Nc}{wc}^{-1}. \tag{5}$$

□

By using Eqs. (3) and (4) in Lemma 3, and the bound on the probability $\Pr[\mu(G) \le L]$, the probability that the minimum span of stopping sets for graph G is less than or equal to L, can be derived. The detailed derivation method is omitted in this paper. For details, see [9].

Theorem 1 ([9]): For $L \in [2, M+1]$, the probability $\Pr[\mu(G) \le L]$ such that $\mu(G)$ is less than or equal to L is bounded by

†"Average" means "average over the ensemble."

$$\begin{aligned}\Pr[\mu(G) \le L] &= \sum_{G\in\mathcal{G}(N,c,d)} \Pr(G) I[\mu(G) \le L] \\ &\le \sum_{G\in\mathcal{G}(N,c,d)} \Pr(G)\Big[I[N_{S_{L,L}}(G) \ge 1] \\ &\quad + \sum_{n=L+1}^{N} I[M_{S_{n,L}}(G) \ge 1]\Big] \\ &= \sum_{w=1}^{L}\left\{1 + \frac{(N-L)w}{L}\right\}\binom{L}{w} W(w). \qquad (6)\end{aligned}$$

□

The bound derived in Theorem 1 assumes the case where code length N is finite. In [9], the case where N is infinite has also been derived.

3. Proposed Code Ensemble

In this section, we will explain the structure of the proposed code which is obtained by concatenating the some parity-check matrices. We mention the concept of the proposed codes after explaining the structure of it. Most of the proposed LDPC codes have good correction capabilities for both random and burst erasures.

3.1 Code Structure

The proposed Tanner graph has three sets of variable nodes V_i, $i = 1, 2, 3$, which are disjoint with each other. We call the sets of variable nodes V_1, V_2, and V_3 the left variable nodes, the middle variable nodes, and the right variable nodes, respectively. The proposed graph also have three sets of edges E_i, $i = 1, 2, 3$, which are disjoint with each other. The set of variable nodes V_i is adjacent to the set of check nodes C with the edge set E_i. Let $G_{\mathrm{LR}} = (V_{\mathrm{LR}} \cup C, E_{\mathrm{LR}})$ be the proposed Tanner graph, where V_{LR} and E_{LR} denote a set of the overall variable nodes and that of edges such that $V_{\mathrm{LR}} = \bigcup_{i=1}^{3} V_i$, $V_{\mathrm{LR}} \cap C = \phi$, and $E_{\mathrm{LR}} = \bigcup_{i=1}^{3} E_i$, respectively.

Each check node of C connects to V_1, V_2, and V_3 with $1, d-2$, and 1 edges, respectively. In other words, the degree of the check nodes C for V_1, V_2, and V_3 are 1, $d-2$, and 1, respectively. Therefore the sum of degree of all the check nodes is $d(= 1 + (d-2) + 1)$. The degrees of V_1, V_2, and V_3 for C are all c. Then the degrees of C for V_{LR} and V_{LR} for C are d and c, respectively, and the overall graph is a (c, d) regular Tanner graph of N variable nodes. Hence the equations $|V_{\mathrm{LR}}| = N$ and $|C| = M$ hold. Since the degree of V_1 is c and the degree of C for V_1 is 1, the number of edges which are incident to V_1 and C, or $|E_1|$, satisfies the equation $|E_1| = |V_1| \times c = |C| \times 1$. Manipulating this equation, we have $|V_1| = \frac{M}{c} = \frac{N}{d}$. In the same way, the cardinalities of V_2 and V_3 are given by $|V_2| = \frac{(d-2)N}{d}$ and $|V_3| = \frac{N}{d}$, respectively. We define $N_1 = |V_1| = |V_3|$, $N_2 = |V_2|$, and $d_j = d - j$ for $j = 2, 3$. The sets of variable nodes are allocated in order of V_1, V_2, and V_3 which means that the equations $V_1 = \{v_1, v_2, \ldots, v_{N_1}\}$, $V_2 = \{v_{N_1+1}, v_{N_1+2}, \ldots, v_{N-N_1}\}$, and $V_3 = \{v_{N-N_1+1}, v_{N-N_1+2}, \ldots, v_N\}$ hold. We call such (c, d) regular Tanner graph the (c, d) left and right (LR) Tanner graph.

The parity-check matrix of the LR Tanner graph, H_{LR}, is represented by

$$H_{\mathrm{LR}} = [H_{\mathrm{LR,lef}} \mid H_{\mathrm{LR,mid}} \mid H_{\mathrm{LR,rig}}], \qquad (7)$$

where $H_{\mathrm{LR,lef}}$, $H_{\mathrm{LR,mid}}$, and $H_{\mathrm{LR,rig}}$ are the $M \times N_1$, $M \times N_2$, and $M \times N_1$ matrices of the row weights 1, $d-2$, and 1, respectively, and of the column weights c. Note that when considering the regularity of the Tanner graph, the parity-check matrix of the code may not have the regularity. Column positions of $H_{\mathrm{LR,lef}}$, $H_{\mathrm{LR,mid}}$, and $H_{\mathrm{LR,rig}}$ correspond to the positions of V_1, V_2, and V_3, respectively.

We then define the ensemble of the LR Tanner graphs in the same way as in Sect. 2.2. For each variable node, we assign c variable sockets. For each check node, we assign three types of check sockets for V_i, $i = 1, 2, 3$. The total number of variable (check) sockets for V_i is therefore $|E_i|$. Define $K_1 = 0$ and $K_i = \sum_{k=1}^{i-1} |E_k|$ for $i = 2, 3$. The LR Tanner graph ensemble is obtained by choosing a permutation function $\pi_i : \{K_i + 1, K_i + 2, \ldots, K_i + |E_i|\} \to \{K_i + 1, K_i + 2, \ldots, K_i + |E_i|\}$ with assigning uniform probability from the space of all permutations of $\{K_i + 1, K_i + 2, \ldots, K_i + |E_i|\} \to \{K_i + 1, K_i + 2, \ldots, K_i + |E_i|\}$ for $i = 1, 2, 3$. For $i = 1, 2, 3$ and for each $j_i = K_i + 1, K_i + 2, \ldots, K_i + |E_i|$, the variable node associated with the j_ith variable socket and the check node associated with the $\pi_i(j_i)$th check socket are adjacent to each other with the edge. Note that there may be the multiple edges which are incident to the same pair of the nodes between V_2 and C. Let $\mathcal{G}_{\mathrm{LR}}(N, c, d)$ be all the collections of the (c, d) LR Tanner graph G_{LR} with N variable nodes.

From the above explanation, since a graph G_{LR} which belongs to $\mathcal{G}_{\mathrm{LR}}(N, c, d)$ is assigned with equal probability, the equality $\Pr(G_{\mathrm{LR}}) = 1/|\mathcal{G}_{\mathrm{LR}}(N, c, d)|$ holds.

It can be easily verified that $\mathcal{G}_{\mathrm{LR}}(N, c, d)$ has the following property.

Lemma 4: For $\mathcal{G}_{\mathrm{LR}}\ (N, c, d)$, we have $|\mathcal{G}_{\mathrm{LR}}\ (N, c, d)| = (N_1c)! \times (N_2c)! \times (N_1c)!$. □

Figure 3 shows (a) the LR Tanner graph ensemble in the same way as in Fig. 1, and (b) its parity-check matrix representation. Note that we abbreviate the sockets for both nodes in Fig. 3(a).

For some subset of bit positions $\mathcal{X} \in [1, N]$, let $G_{\mathrm{LR},\mathcal{X}} = (V_{\mathrm{LR},\mathcal{X}} \cup T(V_{\mathrm{LR},\mathcal{X}}), E_{\mathrm{LR},\mathcal{X}})$ denote a subgraph of the LR Tanner graph G_{LR}.

3.2 Concept of Proposed Code

The difference between the standard Tanner graph and the LR Tanner graph is that the LR Tanner graph have three disjoint sets of variable nodes. Recall that the degree of the check nodes which is adjacent to each of the variable node V_1 or V_3 is one. These sets of variable nodes are useful for correcting the single solid burst erasure. We mention two reasons given by the following theorems:

Theorem 2: Assume that the single solid burst erasure at

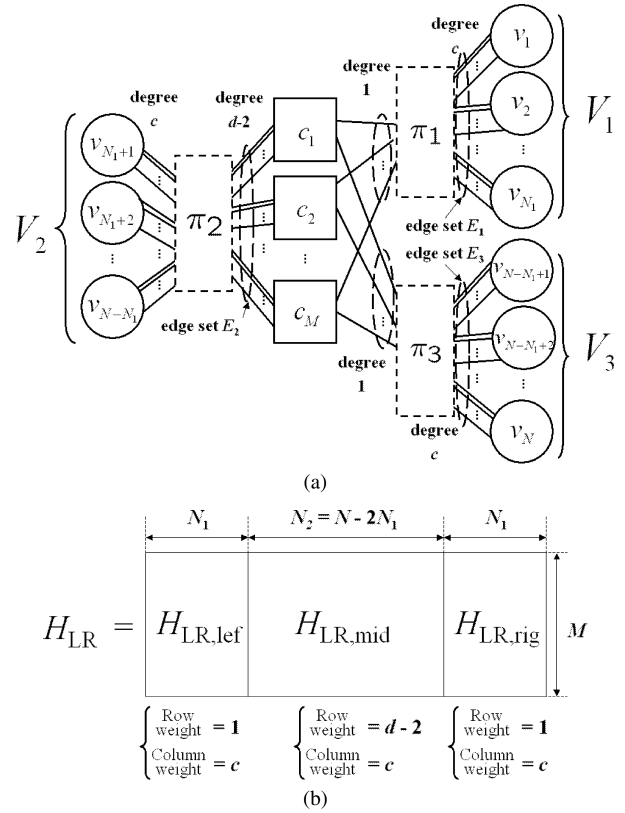

Fig. 3 (a) LR Tanner graph ensemble (b) The parity-check matrix of the LR-LDPC code. Note that when considering the regularity of the Tanner graph, the parity-check matrix of the code may not have the regularity.

the positions in $S_{n,L}$ which covers the positions of the left variable nodes V_1 or those of the right variable nodes V_3 of the LR Tanner graph has been occurred. If this burst erasure covers only V_1 or V_3 which means that $L \le N_1$, then it is guaranteed that this erasure of length L can always be corrected.

Proof: $S_{n,L}$ does not contain any non-empty stopping set, since the degree of all the check nodes of the subgraph $G_{\mathrm{LR},S_{n,L}} = (V_{\mathrm{LR},S_{n,L}} \cup T(V_{\mathrm{LR},S_{n,L}}), E_{\mathrm{LR},S_{n,L}})$ is one. □

For the case where the burst erasure covers both V_1 and V_2 or V_2 and V_3, we give the following theorem by using the idea of Theorem 2:

Theorem 3: Assume that the single solid burst erasure at the positions $S_{n,L}$ which covers the positions of both V_1 and V_2 or those of both V_2 and V_3 of the LR Tanner graph, has occurred. Assume that this burst erasure covers V_1 or V_3 with L_1, $L_1 < L$, erasures and denote the set of this erased bit positions S'. To make the degree of all the check nodes of the subgraph $G_{\mathrm{LR},S_{n,L}} = (V_{\mathrm{LR},S_{n,L}} \cup T(V_{\mathrm{LR},S_{n,L}}), E_{\mathrm{LR},S_{n,L}})$ at least two, or in other words, to make $S_{n,L}$ a stopping set, at least $L_1 (= L - L_1)$ erasures which cover V_2 are needed.

Proof: From the above assumption, the degree of all the check nodes of the subgraph $G_{\mathrm{LR},S'} = (V_{\mathrm{LR},S'} \cup T(V_{\mathrm{LR},S'}), E_{\mathrm{LR},S'})$ are one and the number of these degree one check nodes is $L_1 c$. To make $S_{n,L}$ a stopping set, the degree of these $L_1 c$ check nodes must become at least two. Since the degree of remaining $L - L_1$ variable nodes in V_2 covered by the burst erasure at the positions $S_{n,L}$ are all c, at least L_1 variable nodes at positions $S_{n,L}$ are needed to make $S_{n,L}$ a stopping set. Hence the equality $L_1 = L - L_1$ holds. □

Note that these guarantees given by the above two theorems are not valid for the case of the standard Tanner graphs.

3.3 Average Stopping Set Distribution for LR Tanner Graph Ensemble

In order to derive a bound on the average minimum span of stopping set for the LR Tanner graph ensemble, we first derive the average stopping set distribution for them. In a previous study [8], bounds on the ensemble with two disjoint sets of variable nodes have been derived. The main difference between their ensemble and our ensemble is whether the number of disjoint subset of variable nodes is constant or not.

Let $V'_{\mathrm{LR}} = \bigcup_{i=1}^{3} V'_i \subseteq V_{\mathrm{LR}}$ be a subset of variable nodes of the LR Tanner graph where $V'_i \subseteq V_i$ for $i = 1, 2, 3$. Assume that the degree of all the check nodes in a subgraph $G'_{\mathrm{LR}} = (V'_{\mathrm{LR}} \cup T(V'_{\mathrm{LR}}), E'_{\mathrm{LR}})$ of the graph G_{LR} are at least two. Then the positions of V'_{LR} is a stopping set of size $w = |V'_{\mathrm{LR}}|$ where $w = \sum_{i=1}^{3} w_i$ and $w_i = |V'_{\mathrm{LR}}|$ for $i = 1, 2, 3$. Let $A(w_1, w_2, w_3)$ be the number of stopping sets of a (c, d) LR Tanner graph ensemble with weights w_i, $i = 1, 2, 3$, of the variable nodes V_i such that $0 \le w_1, w_3 \le N_1$ and $0 \le w_2 \le N_2$. Define the function

$$W_{w_1,w_2,w_3} = \binom{N_1 c}{w_1 c}\binom{N_2 c}{w_2 c}\binom{N_1 c}{w_3 c}. \tag{8}$$

$A(w_1, w_2, w_3)$ is obtained from the following lemma.

Lemma 5: The number of stopping set $A(w_1, w_2, w_3)$, of (c, d) LR Tanner graph ensemble is

$$\begin{aligned} A(w_1, w_2, w_3) = {} & \frac{\binom{N_1}{w_1}\binom{N_2}{w_2}\binom{N_1}{w_3}}{W_{w_1,w_2,w_3}} \\ & \times \sum_{c_0,c_1,c_2} \Bigg\{ \binom{M}{c_0, c_1, c_2} \binom{c_1}{w_1 c - c_2} \\ & \times \mathrm{coef}\big[F_{c_0,c_1,c_2}(x), x^{w_2 c}\big] \Bigg\}, \end{aligned} \tag{9}$$

where

$$\begin{aligned} F_{c_0,c_1,c_2}(x) = {} & \Big\{(1+x)^{d_2} - d_2 x\Big\}^{c_0} \\ & \times \Big\{(1+x)^{d_2} - 1\Big\}^{c_1} \Big\{(1+x)^{d_2}\Big\}^{c_2}. \end{aligned} \tag{10}$$

Once w_1, w_2, and w_3 are given in the function W_{w_1,w_2,w_3}, c_2 takes the following range:

$$\kappa_0 \le c_2 \le \min(w_1 c, w_3 c), \tag{11}$$

where κ_0 takes

$$\kappa_0 = \begin{cases} 0, & \text{if } \kappa_1 < 0; \\ \kappa_1, & \text{otherwise,} \end{cases} \tag{12}$$

with $\kappa_1 = (w_1 + w_3)c - M$. Once c_2 is obtained, c_0 and c_1 are determined as follows:

$$c_1 = \frac{(w_1 + w_3)c}{2}, \; c_0 = M - c_1 - c_2. \tag{13}$$

Proof: See Appendix A. □

By using Lemma 5, the total number of stopping set of weight w, $A(w)$, averaged over the LR Tanner graph ensemble (average stopping set distribution) can be easily obtained by summing $A(w_1, w_2, w_3)$ such that $\sum_{i=1}^{3} w_i = w$.

Theorem 4: The average stopping set distribution $A(w)$, $0 \le w \le N$, of (c, d) LR Tanner graph ensemble is

$$A(w) = \sum_{w_1, w_2, w_3} A(w_1, w_2, w_3), \tag{14}$$

where w_0, w_1, and w_2 take $0 \le w_1, w_3 \le N_1$ and $0 \le w_2 \le N_2$ such that $\sum_{i=1}^{3} w_i = w$.

Proof: It is obvious from Lemma 5. □

4. Bound on Average Minimum Span of Stopping Sets for LR Tanner Graph Ensemble

In this section, we derive a bound on the average minimum span of stopping sets for the LR Tanner graph ensemble in the same way as in Sect. 2.5 and by using the method of calculating the average stopping set distribution for the LR Tanner graph ensemble in Sect. 3.3. The result of the bounds where the code length N is finite is described in Sect. 4.1 and it is infinite is described in Sect. 4.2.

4.1 Derivation of Upper Bound

First we formulate a bound on the probability $\Pr[N_{S_{L,L}}(G_{\mathrm{LR}}) \ge 1]$ in the same way as in Lemma 3 where $N_{S_{L,L}}(G_{\mathrm{LR}})$ denote the number of non-empty stopping set $S \in \mathcal{S}(G_{\mathrm{LR}})$ contained in the set of consecutive bit positions $S_{L,L}$ for the (c, d) LR Tanner graph.

The single solid burst erasure at positions $S_{n,L}$ covers some sets of variable nodes V_i for $i = 1, 2, 3$ of the LR Tanner graph. Hence the derivation method of the bound on $\Pr[N_{S_{L,L}}(G_{\mathrm{LR}}) \ge 1]$ depends on L. We divide the range of the value L, $L \in [1, M + 1]$, into the following four cases:

Definition 5: A bound on the probability $\Pr[N_{S_{L,L}}(G_{\mathrm{LR}}) \ge 1]$ is divided into the following four cases which depend on the value of L:

Case 1) $1 \le L \le N_1$
Case 2) $N_1 + 1 \le L \le N - 2N_1$
Case 3) $N - 2N_1 + 1 \le L \le M$
Case 4) $L = M + 1$ □

Then we can bound on $\Pr[N_{S_{L,L}}(G_{\mathrm{LR}}) \ge 1]$ by the following lemma:

Lemma 6: $\Pr[N_{S_{L,L}}(G_{\mathrm{LR}}) \ge 1]$ is bounded by the following equations according to the value of L:

Case 1) For $1 \le L \le N_1$,

$$\Pr[N_{S_{L,L}}(G_{\mathrm{LR}}) \ge 1] \le \sum_{w_1=1}^{L} \frac{\binom{L}{w_1}}{W_{w_1,0,0}} \times \mathrm{coef}[F_{0,w_1c,0}(x), x^0]. \tag{15}$$

Case 2 & 3) For $N_1 + 1 \le L \le M$,

$$\Pr[N_{S_{L,L}}(G_{\mathrm{LR}}) \ge 1] \le \sum_{w=1}^{L} \sum_{\substack{w_1=\max(0,b_1) \\ b_1=N_1-L+w}}^{\min(w,N_1)} \frac{\binom{N_1}{w_1}\binom{L-N_1}{w-w_1}}{W_{w_1,w-w_1,0}} \times \binom{M}{w_1c} \times \mathrm{coef}[F_{M-w_1c,w_1c,0}(x), x^{(w-w_1)c}]. \tag{16}$$

Case 4) For $L = M + 1$,

$$\Pr[N_{S_{L,L}}(G_{\mathrm{LR}}) \ge 1] \le \sum_{w=1}^{L} \sum_{\substack{w_1=\max(0,b_1) \\ b_1=w-N_2-L-1}}^{\min(w,N_1)} \sum_{\substack{w_3=\max(0,b_2) \\ b_2=w-N_1-N_2}}^{1} \frac{\binom{N_1}{w_1}\binom{N_2}{w-w_1-w_3}\binom{1}{w_3}}{W_{w_1,w-w_1-w_3,w_3}} \times \sum_{c_0,c_1,c_2} \left\{ \binom{M}{c_0, c_1, c_2} \binom{c_1}{w_1c - c_2} \times \mathrm{coef}[F_{c_0,c_1,c_2}(x), x^{(w-w_1-w_3)c}] \right\}. \tag{17}$$

where c_0, c_1, and c_2 are obtained by using Eqs. (11)–(13).

Proof: See Appendix B. □

Next we derive an upper bound on the probability $\Pr[M_{S_{n,L}}(G_{\mathrm{LR}}) \ge 1]$ in the same way as in Lemma 3 where $M_{S_{n,L}}(G_{\mathrm{LR}})$ denotes the number of non-empty stopping set $S \in \mathcal{S}(G_{\mathrm{LR}})$ included in the set of positions $S_{n,L}$ with a position n which is included in S for a graph G_{LR}. In the similar manner as in Definition 5, the derivation method of the bound on $\Pr[M_{S_{n,L}}(G_{\mathrm{LR}}) \ge 1]$ depends on L and n, and each case given in Definition 5 is divided into some number of cases which depend on the value of n.

Definition 6: A bound on the probability $\Pr[M_{S_{n,L}}(G_{\mathrm{LR}}) \ge 1]$ is divided into the following cases which depend on the values of L and n:

Case 1) The set of bit positions is divided into five cases such as (a) $n \in [L+1, N_1]$, (b) $n \in [N_1+1, N_1+L-1]$, (c) $n \in [N_1+L, N-N_1]$, (d) $n \in [N-N_1+1, N-N_1+L-1]$, and (e) $n \in [N-N_1+L, N]$.
Case 2) The set of bit positions is divided into three cases such as (a) $n \in [L+1, L+N_1-1]$, (b) $n \in [L+N_1, N-N_1]$, and (c) $n \in [N-N_1+1, N]$.
Case 3) The set of bit positions is divided into three cases such as (a) $n \in [L+1, N-N_1]$, (b) $n \in [N-N_1+1, L+N_1-1]$, and (c) $n \in [L+N_1, N]$.

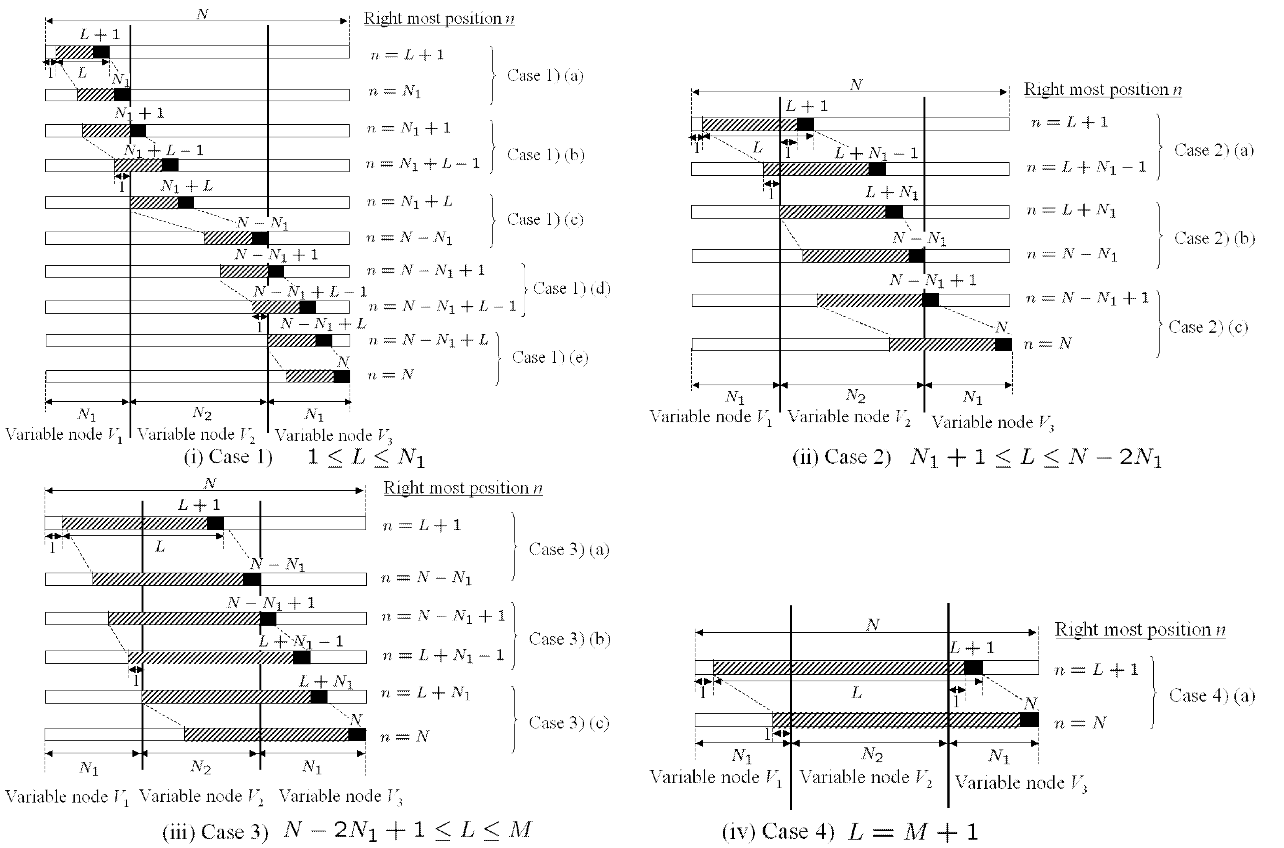

Fig. 4 The range of a solid burst erasure $S_{n,L}$ given by Definition 6 for various values of n and L. The shadow and filled black box indicate the burst erasure and the filled black box represents the bit position n which is the rightmost one of $S_{n,L}$. (i) For Case 1) (a)–(e) (ii) For Case 2) (a)–(c) (iii) For Case 3) (a)–(c) (iv) For Case 4) (a).

Case 4) In case 4), there is only one case such as (a) $n \in [L+1, N]$. □

The schematic description for each case in Definition 6 is depicted in Fig. 4. For the various values of L and n, this figure shows the positions of the range of some burst erasures at $S_{n,L}$. The length of all the ranges is L and is represented by the shadow and black box, where the black box indicates the rightmost bit position n. For various values of L, we can divide into four cases 1)–4). Moreover each case is divided into some number of cases for various n. For example, the Case 1) shown in Fig. 4 (i) is divided into five cases (a)–(e).

To derive a bound on the probability $\Pr[M_{S_{n,L}}(G_{\mathrm{LR}}) \geq 1]$, showing bounds for all the cases mentioned above is needed. Only the case 2) is important to bound on $\Pr[M_{S_{n,L}}(G_{\mathrm{LR}}) \geq 1]$, and the other cases are meaningless to bound on $\Pr[M_{S_{n,L}}(G_{\mathrm{LR}}) \geq 1]$. For the case 1), the value of L is too small for the burst erasure correction capability of the LDPC codes. For the case 4), since all linear block codes cannot correct $M+1$ or larger erasures, we do not need to bound on $\Pr[M_{S_{n,L}}(G_{\mathrm{LR}}) \geq 1]$. For the case 3), we prove by the following lemma.

Lemma 7: Only $(c, c+1)$ LR Tanner graphs satisfy the case 3) in Definition 5.

Proof: From the definition of the (c, d) Tanner graphs, the parameters c and d satisfy $d > c$. Noting that the condition of the case 3) in Definition 5 is expressed as $N - 2N_1 < M$, we have $d_2 < c$ since $N - 2N_1 = N - 2 \times \frac{N}{d} = \frac{d_2}{d}$ and $M = \frac{c}{d} \times N$. Combining $d > c$ and $d_2 < c$ gives $d_2 < c < d$ and we obtain $d = c + 1$. Therefore there exists only one d which satisfies the above equation and we obtain $d = c + 1$. □

The number of nodes in the middle variable nodes V_2, $\frac{d_2}{d} \times N$, is always lower than the number of check nodes $\frac{c}{d} \times N (= M)$. Therefore $(c, c+1)$ LR Tanner graph codes are not useful for the burst erasures which cover all the bit positions of V_2.

From the above reasons, we show the bound on $\Pr[M_{S_{n,L}}(G_{\mathrm{LR}}) \geq 1]$ for only the case 2) in the following corollary.

Corollary 1: For the case 2), the probability $\Pr[M_{S_{n,L}}(G_{\mathrm{LR}}) \geq 1]$ is bounded by the following equations according to the value of n:

Case 2-(a)) For $n \in [L+1, L+N_1-1]$,

$$
\begin{aligned}
&\Pr[M_{S_{n,L}}(G_{\mathrm{LR}}) \geq 1] \\
&\leq \sum_{w=1}^{L} \sum_{\substack{w_1=\max(0,b_1) \\ b_1=w-(n-N_1)}}^{\substack{\min(w,b_2) \\ b_2=L-(n-N_1)}} \frac{\binom{L-(n-N_1)}{w_1}\binom{n-N_1-1}{w-w_1-1}}{W_{w_1,w-w_1,0}} \binom{M}{w_1 c} \\
&\quad \times \mathrm{coef}[F_{M-w_1c,w_1c,0}(x), x^{(w-w_1)c}]. \qquad (18)
\end{aligned}
$$

Case 2-(b)) For $n \in [L + N_1, N - N_1]$,

$$
\begin{aligned}
&\Pr[M_{S_{n,L}}(G_{\mathrm{LR}}) \geq 1] \\
&\leq \sum_{w_2=1}^{L} \frac{\binom{L-1}{w_2-1}}{W_{0,w_2,0}} \times \mathrm{coef}[F_{M,0,0}(x), x^{w_2 c}]. \qquad (19)
\end{aligned}
$$

Case 2-(c)) For $n \in [N - N_1 + 1, N]$,

$$
\begin{aligned}
&\Pr[M_{S_{n,L}}(G_{\mathrm{LR}}) \geq 1] \\
&\leq \sum_{w=1}^{L} \sum_{\substack{w_3=\max(0,b_1) \\ b_1=w-L+n-(N-N_1)}}^{\substack{\min(w,b_2) \\ b_2=n-(N-N_1)}} \frac{\binom{L-n+(N-N_1)}{w-w_3}\binom{n-(N-N_1)-1}{w_3-1}}{W_{0,w-w_3,w_3}} \\
&\quad \times \binom{M}{w_3 c} \times \mathrm{coef}[F_{M-w_3c,w_3c,0}(x), x^{(w-w_3)c}]. \qquad (20)
\end{aligned}
$$

□

From Eq. (6), the same argument holds for the case of LR Tanner graph ensemble. Then we substitute G_{LR} into G and $\mathcal{G}_{\mathrm{LR}}(N, c, d)$ into $\mathcal{G}(N, c, d)$ in Eq. (6), obtaining

$$
\begin{aligned}
&\Pr[\mu(G_{\mathrm{LR}}) \leq L] \\
&\leq \sum_{G_{\mathrm{LR}} \in \mathcal{G}_{\mathrm{LR}}(N,c,d)} \Big[\Pr(G_{\mathrm{LR}}) I[\{N_{S_{L,L}}(G_{\mathrm{LR}}) \geq 1\}] \\
&\quad + \sum_{n=L+1}^{N} I[\{M_{S_{n,L}}(G_{\mathrm{LR}}) \geq 1\}] \Big]. \qquad (21)
\end{aligned}
$$

Substituting Eqs. (16) and (18)–(20) to Eq. (21) by the similar way as in Theorem 1, we have the following theorem.

Theorem 5: For the case 2), the average minimum span of stopping sets for the (c, d) LR Tanner graph ensemble is bounded by

$$
\begin{aligned}
&\Pr[\mu(G_{\mathrm{LR}}) \leq L] \\
&\leq \sum_{w=1}^{L} \Bigg\{ \frac{(N - 2N_1 - L + 1)w + 1}{L} \times \frac{\binom{L}{w}}{W_{0,w,0}} \\
&\quad \times \mathrm{coef}[F_{M,0,0}(x), x^{wc}] \Bigg\} \\
&\quad + 2 \times \sum_{w=1}^{L} \sum_{w_1=1}^{\min(w,N_1)} \sum_{k=w_1}^{N_1-1} \Bigg\{ \frac{\binom{k-1}{w_1-1}\binom{L-k}{w-w_1}}{W_{w_1,w-w_1,0}} \\
&\times \binom{M}{w_1 c} \times \mathrm{coef}[F_{M-w_1c,w_1c,0}(x), x^{(w-w_1)c}] \Bigg\}. \qquad (22)
\end{aligned}
$$

Proof: Substituting Eqs. (16), (18), (19), and (20) into Eq. (21) gives Eq. (22). We also substitute w into w_2 in Eq. (19) and w into w_3 in Eq. (20). The first term of r.h.s. of Eq. (22) is the sum of Eq. (16) and Eq. (18) when $w_1 = 0$. The second term of r.h.s. of this equation is the sum of Eq. (18) when $w_1 > 0$ and Eqs. (19) and (20). □

4.2 Asymptotic Analysis for LR Tanner Graph Ensemble

In this section, we derive the asymptotic average value of the minimum span of stopping sets for the LR Tanner graph ensemble, which enables us to analyze asymptotic performance for a solid burst erasure.

Theorem 6: For the case 2) and for a (c, d) LR-LDPC code ensemble, the inequality

$$
\lim_{N\to\infty} \frac{1}{N} \log \Pr[\mu(G_{\mathrm{LR}}) \leq \omega N] \leq B_{\mathrm{LR}}(\omega), \qquad (23)
$$

holds where $B_{\mathrm{LR}}(\omega)$ is defined by

$$
\begin{aligned}
B_{\mathrm{LR}}(\omega) = &\sup_{0<\alpha\leq\omega} \Bigg[\omega H\left(\frac{\alpha}{\omega}\right) + \sup_{0\leq\alpha_1\leq\min(\alpha,\frac{1}{d})} [(1 - R') \\
&\times \left\{ H\left(\frac{\alpha_1 c}{1-R'}\right) - H(d\alpha_1) - d_2 H\left(\frac{d(\alpha-\alpha_1)}{d_2}\right)\right\} \\
&+ \log\left(\frac{F_{1-R'-\alpha_1 c,\alpha_1 c,0}(x_0)}{x_0^{(\alpha-\alpha_1)c}}\right)\Bigg]\Bigg], \qquad (24)
\end{aligned}
$$

such that $2\alpha_1 \leq \alpha$ and $0 \leq \alpha_1 \leq \frac{1}{d}$. x_0 is a unique positive solution of

$$
\begin{aligned}
&\frac{(1 - R' - \alpha_1 c)d_2 x\{(1 + x)^{d_3} - 1\}}{(1 + x)^{d_2} - d_2 x} \\
&+ \frac{\alpha_1 c d_2 x (1 + x)^{d_3}}{(1 + x)^{d_2} - 1} = (\alpha - \alpha_1)c. \qquad (25)
\end{aligned}
$$

Proof: See Appendix C. □

5. Code Obtained by Column Permutation

The present authors have proposed a column permutation algorithm for a parity-check matrix H which can improve the performance for multiple solid burst erasures [11]. Let $\tilde{H}$ denote the column permuted parity-check matrix of H. We now make the leftmost (rightmost) position of element one at each row of $\tilde{H}$ as a small (large) value as possible. Assume that $\tilde{H}$ has the following form:

$$
\tilde{H} = [\tilde{H}_{\mathrm{lef}}, \tilde{H}_{\mathrm{mid}}, \tilde{H}_{\mathrm{rig}}]. \qquad (26)
$$

where $\tilde{H}_{\mathrm{lef}}$ and $\tilde{H}_{\mathrm{rig}}$ are $M \times (N_1 - r')$ and $M \times (N_1 - r'')$, $0 \leq r', r'' < N_1$, matrices. The weights of r' (r'') rows and $M - r'$ ($M - r''$) rows of these matrices are zero and one, respectively, and the weights of all the columns of $\tilde{H}$ are c.

The structure of the column permuted parity-check matrix $\tilde{H}$ given by Eq. (26) is similar to that of the LR-LDPC codes when r' and r'' take small values. The set of variable nodes V_1, V_2, and V_3 of the LR-LDPC codes correspond to the set of variable nodes of $\tilde{H}_{\rm lef}$, $\tilde{H}_{\rm mid}$, and $\tilde{H}_{\rm rig}$ in its Tanner graphs, respectively. Therefore we can obtain a code having a similar structure to the LR-LDPC codes by a column permutation algorithm [11]†. We call this code the pseudo LR-LDPC (PLR-LDPC) code and denote its Tanner graph by $G_{\rm PLR}$.

6. Some Examples and Discussions

In order to show the effectiveness of the proposed Tanner graph ensemble, we compare codes obtained by ensemble analysis with those generated by the computer.

6.1 Numerically Calculated Examples

6.1.1 Critical Minimum Span Ratio of Stopping Sets

Tables 1 and 2 show the critical minimum span ratio of the stopping sets $\omega^\star$ for the (c, d) standard Tanner graph (Standard) [9] and that for the (c, d) LR Tanner graph (LR) ensembles. The value $\omega^\star$ for LR Tanner graph ensemble is obtained by calculating

$$\omega^\star = \inf_{\substack{0<\omega\le 1-R' \\ B_{\rm LR}(\omega)\ge 0}} \omega. \tag{27}$$

Tables 1 and 2 show the cases where $c = 3$ and where $R' = 0.5$, respectively. From Table 1, $\omega^\star$ for the LR Tanner graph ensembles are larger than those for the standard Tanner graph ensembles in the cases of the $(3, 5)$, $(3, 6)$, $(3, 9)$, and $(3, 12)$ Tanner graph. For the $(3, 4)$ Tanner graph, $\omega^\star$ for the LR Tanner graph ensemble is smaller than that for the standard ones, since the code with $(c, d) = (3, 4)$ satisfies $d = c + 1$, which is not good for the burst correction capability as mentioned in Sect. 4.1. In Table 2, these values for the LR Tanner graphs are slightly larger than those for the standard Tanner graphs in the case of the $(2, 4)$ Tanner graphs and these two values are the same for the $(4, 8)$ Tanner graphs. These values for the LR Tanner graph ensembles are slightly smaller than those for the standard one in the cases of $(5, 10)$ and $(6, 12)$ Tanner graphs.

6.1.2 Critical Exponent of Stopping Ratio

We compare the critical exponent of stopping ratio $\alpha^\star$ for the standard Tanner graph ensemble [7] and the LR Tanner graph ensemble. The value $\alpha^\star$ expresses that the probability of existence of the stopping set weight equal to or less than $\alpha^\star \times N$ goes to 0 as code length N tends to infinity. Therefore $\alpha^\star$ is one of measures for random erasure correction capability of the Tanner graphs by the BP decoding algorithm.

For a rational number $\alpha > 0$, the critical exponent of stopping ratio $\alpha^\star$ for the LR Tanner graph ensemble can be obtained by

$$\alpha^\star = \inf_{\substack{0<\alpha\le 1-R' \\ \mathcal{A}(\alpha)\ge 0}} \alpha, \tag{28}$$

Table 1 Critical minimum span ratio of stopping sets $\omega^\star$ for two Tanner graph ensembles when $c = 3$.

c	d	R'	Standard	LR
3	4	0.25	0.571	0.5
3	5	0.4	0.447	0.457
3	6	0.5	0.366	0.373
3	9	0.667	0.237	0.239
3	12	0.75	0.175	0.176

Table 2 Critical minimum span ratio of stopping sets $\omega^\star$ for two Tanner graph ensembles when $R' = 0.5$.

c	d	R'	Standard	LR
2	4	0.5	0.326	0.369
3	6	0.5	0.366	0.373
4	8	0.5	0.324	0.324
5	10	0.5	0.286	0.285
6	12	0.5	0.256	0.255

Table 3 Average value (Ave.) and its standard deviations (S. Dv.) of the minimum span of stopping sets $\mu(G)$, $\mu(G_{\rm LR})$, and $\mu(G_{\rm PLR})$ for the graphs with $N = 1008$, $c = 3$, and $d = 6$. The number of samples for each graph is 20.

	$\mu(G)$	$\mu(G_{\rm LR})$	$\mu(G_{\rm PLR})$
Ave.	404.7	413.65	414.95
S. Dv.	4.28	3.92	4.80

where $\mathcal{A}(\alpha)$ is given by

$$\mathcal{A}(\alpha) = \lim_{N\to\infty} \frac{1}{N} \log A(\alpha N). \tag{29}$$

The values $\alpha^\star$ for both (3,6) Tanner graph ensembles are 0.018. For other pairs of parameters (c, d) in Tables 1 and 2, we confirm that they have also the same values. This implies that both code ensemble have the same random erasure correction capability from the view-point of critical exponent of stopping ratio.

6.2 Experimental Results

We compare the average minimum span of stopping sets for the Tanner graphs G, $G_{\rm LR}$, and $G_{\rm PLR}$ by using the graphs generated by the computer. We generate 20 samples for each graph of length $N = 1008$, $c = 3$, and $d = 6$ by using the different seeds of random generators. We evaluate the values of average minimum span of stopping sets $\mu(G)$, $\mu(G_{\rm LR})$, and $\mu(G_{\rm PLR})$ by executing the $L_{\max}$ algorithm [4], which calculates the r.h.s. of Eq. (1). Table 3 shows these values averaged over 20 samples where "Ave." and "S. Dv." denote the average values of minimum span and its standard deviations, respectively. From this table, the values of $\mu(G_{\rm LR})$ and $\mu(G_{\rm PLR})$ are slightly larger than those of $\mu(G)$.

6.3 Discussions

From Tables 1–3, the burst erasure correction capabilities

†To obtain $\tilde{H}$ in Eq. (26), we may only perform step (A) and (B) of the algorithm in [11].

for the LR Tanner graphs are larger than that for the standard Tanner graphs. By column permutation for a parity-check matrix of the standard LDPC codes to be close to a parity-check matrix of the LR-LDPC codes, we obtain PLR-LDPC codes having almost the same performance for the burst erasure correction. We have confirmed that the graphs of the LR-LDPC codes and that of the PLR-LDPC codes are statistically meaningful by executing the hypothesis test for the average values compared with the graphs of the standard LDPC codes. Moreover (c, d) LR Tanner graphs with $d > c + 1$ can correct almost equal or a longer solid burst erasure than the standard ones without a degradation in performance for the random erasures in Sect. 6.1.2.

From Table 3, the ratio of the average minimum span to code length for the standard Tanner graphs $\frac{\mu(G)}{N}$ and LR Tanner graphs $\frac{\mu(G_{\mathrm{LR}})}{N}$ are approximately 0.401 and 0.410, respectively, and they are larger than the upper bounds presented in Tables 1 and 2. This is because we have selected the graphs having high performance generated by the computer, even though there exists the bad ones. The tightness of the bounds derived in this paper depends on the Markov's inequality need in Eqs. (3) and (4), and therefore there is a gap between bounds and experiments for both Tanner graphs.

7. Conclusion and Further Works

We have proposed a new regular Tanner graph ensemble suitable for correcting a solid burst erasure. From the numerically calculated bound on the asymptotic capability for single solid burst erasure correction, the proposed Tanner graph ensemble are better than that of the conventional ones for most of the parameters of the codes. From experimental results, we also show that the average minimum span of stopping sets for the proposed ensemble is larger than that of the conventional ones.

Though (c, d) LR Tanner graph ensembles are a subclass of regular ones, it is worthy noting that instead of using the irregular Tanner graphs, we obtain a new regular Tanner graph ensemble suitable for correcting a solid burst erasure.

For further works, analysis of the proposed codes for multiple solid burst erasures is needed. Analysis of the LR type irregular Tanner graphs is also remained.

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References

[1] R.G. Gallager, "Low density parity check codes," IRE Trans. Inf. Theory, vol.8, pp.21–28, Jan. 1962.

[2] M. Luby, M. Mitzenmacher, M.A. Shokrollahi, and D.A. Spielman, "Efficient erasure correcting codes," IEEE Trans. Inf. Theory, vol.47, no.2, pp.569–584, Feb. 2001.

[3] C. Di, D. Proietti, I.E. Telatar, T.J. Richardson, and R.L. Urbanke, "Finite-length analysis of low-density parity-check codes on the binary erasure channel," IEEE Trans. Inf. Theory, vol.48, no.6, pp.1570–1579, June 2002.

[4] M. Yang and W.E. Ryan, "Performance of efficiently encordable low-density parity-check codes in noise bursts on the EPR4 channel," IEEE Trans. Magn., vol.40, no.2, pp.507–512, March 2004.

[5] D. Burshtein and G. Miller, "Asymptotic enumeration methods for analyzing LDPC codes," IEEE Trans. Inf. Theory, vol.50, no.6, pp.1115–1131, June 2004.

[6] T. Wadayama, "Greedy construction of LDPC codes for burst erasure channel," IEICE Technical Report, IT2004-44, Sept. 2004.

[7] A. Oritsky, K. Viswanathan, and J. Zhang, "Stopping set distribution of LDPC code ensemble," IEEE Trans. Inf. Theory, vol.51, no.3, pp.929–953, March 2005.

[8] R. Ikegaya, K. Kasai, Y. Shimoyama, T. Shibuya, and K. Sakaniwa, "Weight and stopping set distributions of two-edge type LDPC code ensemble," IEICE Trans. Fundamentals, vol.E88-A, no.10, pp.2745–2761, Oct. 2005.

[9] T. Wadayama, "Ensemble analysis on minimum span of stopping sets," Proc. Information Theory and Its Applications Workshop, UCSD, Feb. 2006.

[10] E. Paolini and M. Chiani, "Improved low-density parity-check dodes for burst erasure channels," Proc. 2006 IEEE International Conference on Commun. (ICC06), Istanbul, June 2006.

[11] G. Hosoya, H. Yagi, T. Matsushima, and S. Hirasawa, "A modification method for constructing low-density parity-check codes for burst erasures," IEICE Trans. Fundamentals, vol.E89-A, no.10, pp.2501–2509, Oct. 2006.

[12] T. Wadayama, "Average coset weight distributions of combined LDPC matrix ensembles," IEEE Trans. Inf. Theory, vol.52, no.11, pp.4856–4866, Nov. 2006.

Appendix A: Proof of Lemma 5

We first derive Eq. (9) and then derive Eqs. (11)–(13).

(1) Derivation of Eq. (9)

Assume that choosing some set of variable nodes V'_{LR} with arbitrary size $w = |V'_{\mathrm{LR}}|$ from the graph G_{LR} and construct its subgraph $G'_{\mathrm{LR}} = (V'_{\mathrm{LR}} \cup T(V'_{\mathrm{LR}}), E'_{\mathrm{LR}})$. We derive the probability that the set of bit positions of V'_{LR} has a stopping set. For $i = 1, 2, 3$ such that $\sum_{i=1}^{3} w_i = w$, recall that $A(w_1, w_2, w_3)$ is the number of stopping sets of weight w_i choosing w_i nodes from V_i. We first count the number of ways of choosing w_i variable nodes from the variable nodes V_i. Then multiplying these combinations gives

$$D_1 = \binom{N_1}{w_1}\binom{N_2}{w_2}\binom{N_1}{w_3}. \tag{A·1}$$

In a similar manner, we count the number of ways of choosing $w_i c$ edges from each set of $N_1 c$, $N_2 c$, and $N_1 c$ variable node edges in the graph G_{LR}. Multiplying these combinations gives

$$D_2 = \binom{N_1 c}{w_1 c}\binom{N_2 c}{w_2 c}\binom{N_1 c}{w_3 c}. \tag{A·2}$$

Dividing D_1 with D_2 in Eqs. (A·1) and (A·2) such that $D_3 = D_1/D_2$, we can obtain the probability of choosing the subgraph G'_{LR} from the graph G_{LR}.

After choosing w_1 variable nodes from V_1 and w_3 variable nodes from V_3, the degrees of check nodes C which are adjacent to V_1 and V_3 take zero, one, or two. We define the numbers of these check nodes as c_0, c_1, and c_2 such that

$$c_0 + c_1 + c_2 = M. \tag{A·3}$$

The derivation method of c_0, c_1, and c_2 will be discussed in this appendix (2). We count the ways of choosing these c_0, c_1, and c_2 check nodes from M check nodes as

$$D_4 = \binom{M}{c_0, c_1, c_2}. \tag{A·4}$$

Moreover c_1 degree one check nodes are either adjacent to V_1 or V_3. We count the number of way of choosing the degree one check node which is adjacent to V_1 (or equivalently to V_3) from c_1 check nodes. There are w_1c edges are incident to V_1 and to C. These edges are incident to either degree one or degree two check nodes. Since the number of degree two check nodes is c_2, the number of degree one check nodes which are adjacent to V_1 is $w_1c - w_2$. Therefore

$$D_5 = \binom{c_1}{w_1c - c_2}. \tag{A·5}$$

Multiplying D_4 and D_5 in Eqs. (A·4) and (A·5) such that $D_6 = D_4 \times D_5$ gives the check nodes selection after choosing w_1 and w_3 variable nodes.

Finally we count the number of edge connections from C to V_2 such that the degrees of all the check nodes become zero or greater than one to satisfy a condition of the stopping set. The number of these edges is w_2c. By using the following generating function

$$\begin{aligned} D_7 = \mathrm{coef}[&\{(1+x)^{d_2} - d_2x\}^{c_0} \\ &\times \{(1+x)^{d_2} - 1\}^{c_1}\{(1+x)^{d_2}\}^{c_2}, x^{w_2c}], \end{aligned} \tag{A·6}$$

to retrieve the coefficient of the term x^{w_2c}, we evaluate this combination of edge connections.

Multiplying D_6 and D_7 and combining with D_3, we obtain Eq. (9).

(2) Derivation of Eqs. (11)–(13)

We first derive Eq. (11). From the above discussion, the equality

$$c_1 + 2c_2 = (w_1 + w_3)c \tag{A·7}$$

holds between c_1 and c_2, since $c_1 + c_2$ check nodes are adjacent to V_1 or V_3 with the $c_1 + 2c_2$ edges from C to V_1 and V_3 or equivalently with $(w_1 + w_3)c$ edges from V_1 and V_3 to C. There are c_2 check nodes which are adjacent to both V_1 and V_3 with the edges. For each check node, there are two edges; one edge for V_1 and another for V_3. Since the number of edges which are incident to V_1 and to C is w_1c and to V_3 and to C is w_3c, c_2 takes at most $\min(w_1c, w_3c)$. The possibility of the subgraph having check nodes either adjacent to V_1 or V_3 is at most $(w_1+w_3)c$, but the number of check nodes is at most M. If the inequality $(w_1+w_3)c - M < 0$ holds, and c_2 takes 0. Otherwise $(w_1+w_3)c - M$ check nodes which are adjacent to both V_1 and V_3, then c_2 takes $c_2 = (w_1+w_3)c - M$. Therefore we obtain Eq. (11).

Substituting c_2 to Eq. (A·7), we obtain c_1 in Eq. (13). Substituting c_1 and c_2 to Eq. (A·3), we obtain c_0 in Eq. (13). □

Appendix B: Proof of Lemma 6

In the similar manner as in Appendix A, assume that choosing some consecutive set of the variable nodes $S_{L,L}$ from the graph G_{LR} and construct its subgraph $G_{\mathrm{LR},S_{L,L}} = (V_{\mathrm{LR},S_{L,L}} \cup T(V_{\mathrm{LR},S_{L,L}}), E_{\mathrm{LR},S_{L,L}})$. For the cases 2) and 3) where L satisfies $N_1 \leq L \leq M$, the single solid burst erasure at $S_{L,L}$ which covers the set of variable nodes V_1 and V_2, and the equality $w_3 = 0$ is always satisfied. We first count the number of ways of choosing w_1 variable nodes from V_1 and $w - w_1 (= w_2)$ variable nodes from V_2. We then count the overall edge selection, the number of ways of choosing w_1c edges among N_1c edges and $(w - w_1)c$ edges among N_2c edges. After choosing w_1 variable nodes from V_1 and in a similar manner, choosing edges from V_1, the degrees of all the check nodes C take either zero or one. Since there are w_1c degree one check nodes and other $M - w_1c$ degree zero check nodes, the equalities $c_0 = M - w_1c$, $c_1 = w_1c$, and $c_2 = 0$ hold. We count the ways of choosing these c_0, c_1, and c_2 check nodes from M check nodes and we obtain $\binom{M}{w_1c}$. Since all c_1 degree one check nodes which are adjacent to only V_1 and are not adjacent to V_3, we do not need to count this check node selection. From the above discussion, we obtain

$$\frac{\binom{N_1}{w_1}\binom{L-N_1}{w-w_1}\binom{M}{w_1c}}{\binom{N_1c}{w_1c}\binom{N_2c}{(w-w_1)c}}. \tag{A·8}$$

Finally, the number of ways for connecting $(w - w_1)c$ edges which are incident to C and V_2, we use the following generating function:

$$\begin{aligned} \mathrm{coef}[&\{(1+x)^{d_2} - d_2x\}^{M-w_1c} \\ &\times\{(1+x)^{d_2} - 1\}^{w_1c}, x^{(w-w_1)c}], \end{aligned} \tag{A·9}$$

to retrieve the coefficient of the term $x^{(w-w_1)c}$. The range of the value of w_1 can be easily derived from the fact $0 \leq w_1 \leq N_1$ and using the hyper-geometric distribution. Multiplying Eqs. (A·8) and (A·9) to bound in a similar manner as Eq. (3), we obtain Eq. (16).

By a similar argument, Eqs. (15) and (17) can also be derived. □

Appendix C: Proof of Theorem 6

Let $L = \omega N$, $w = \alpha N$, and $w_1 = \alpha_1 N$, where ω, α, and α_1 denote the rational numbers. We want to derive an upper bound on

$$\lim_{N\to\infty}\frac{1}{N}\log\Pr[\mu(G_{\mathrm{LR}})\le\omega N]. \tag{A· 10}$$

We can obtain Eq. (24) from Eq. (22) by the following step.

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\log\Pr[\mu(G_{\mathrm{LR}})\le\omega N]\\
&\le\lim_{N\to\infty}\frac{1}{N}\log\left[\sum_{w=1}^{\omega N}\left\{\frac{(N-2N_1-\omega N+1)w+1}{\omega N}\right.\right.\\
&\left.\times\frac{\binom{\omega N}{w}}{W_{0,w,0}}\times\mathrm{coef}[F_{M,0,0}(x),x^{wc}]\right\}\\
&+2\sum_{w=1}^{\omega N}\sum_{w_1=1}^{\min(w,N_1)}\sum_{k=w_1}^{N_1-1}\left\{\frac{\binom{k-1}{w_1-1}\binom{\omega N-k}{w-w_1}}{W_{w_1,w-w_1,0}}\right.\\
&\left.\left.\times\binom{M}{w_1c}\times\mathrm{coef}[F_{M-w_1c,w_1c,0}(x),x^{(w-w_1)c}]\right\}\right]\\
&=\lim_{N\to\infty}\frac{1}{N}\log(B_1+B_2),
\end{aligned}\tag{A· 11}$$

where B_1 and B_2 correspond to the first and second terms of r.h.s. in Eq. (A· 11). Next we bound on B_1 and B_2 by the following (1) and (2).

(1) Bound on B_1;

$$\begin{aligned}
B_1&=\sum_{w=1}^{\omega N}\left\{\frac{(N-2N_1-\omega N+1)w+1}{\omega N}\right.\\
&\left.\quad\times\frac{\binom{\omega N}{w}}{W_{0,w,0}}\times\mathrm{coef}[F_{M,0,0}(x),x^{wc}]\right\}\\
&=\left(N-2N_1-\omega N+1+\frac{1}{\omega N}\right)\\
&\quad\times\sum_{w=1}^{\omega N}\left\{\frac{\binom{\omega N}{w}}{W_{0,w,0}}\times\mathrm{coef}[F_{M,0,0}(x),x^{wc}]\right\}\\
&\le\left\{(N-2N_1-\omega N+1)\omega N+1\right\}\\
&\quad\times\max_{1\le w\le\omega N}\left\{\frac{\binom{\omega N}{w}}{W_{0,w,0}}\times\mathrm{coef}[F_{M,0,0}(x),x^{wc}]\right\}.
\end{aligned}\tag{A· 12}$$

(2) Bound on B_2;

$$\begin{aligned}
B_2&=2\sum_{w=1}^{\omega N}\sum_{w_1=1}^{\min(w,N_1)}\sum_{k=w_1}^{N_1-1}\left\{\frac{\binom{k-1}{w_1-1}\binom{\omega N-k}{w-w_1}}{W_{w_1,w-w_1,0}}\right.\\
&\quad\left.\times\binom{M}{w_1c}\times\mathrm{coef}[F_{M-w_1c,w_1c,0}(x),x^{(w-w_1)c}]\right\}\\
&=2\sum_{w=1}^{\omega N}\sum_{w_1=1}^{\min(w,N_1)}\left[\frac{\mathrm{coef}[F_{M-w_1c,w_1c,0}(x),x^{(w-w_1)c}]}{W_{w_1,w-w_1,0}}\right.\\
&\quad\left.\times\binom{M}{w_1c}\sum_{k=w_1}^{N_1-1}\binom{k-1}{w_1-1}\binom{\omega N-k}{w-w_1}\right]\\
&\le2\sum_{w=1}^{\omega N}\left[\binom{\omega N}{w}\sum_{w_1=1}^{\min(w,N_1)}\binom{M}{w_1c}\right.\\
&\quad\left.\times\frac{\mathrm{coef}[F_{M-w_1c,w_1c,0}(x),x^{(w-w_1)c}]}{W_{w_1,w-w_1,0}}\right]\\
&\le2\sum_{w=1}^{\omega N}\left[\binom{\omega N}{w}\times\min(w,N_1)\max_{\substack{1\le w_1\le b_3\\ b_3=\min(w,N_1)}}\left\{\binom{M}{w_1c}\right.\right.\\
&\quad\left.\left.\times\frac{\mathrm{coef}[F_{M-w_1c,w_1c,0}(x),x^{(w-w_1)c}]}{W_{w_1,w-w_1,0}}\right\}\right]\\
&\le2\sum_{w=1}^{\omega N}\left[\binom{\omega N}{w}\times\omega N\max_{\substack{1\le w_1\le b_3\\ b_3=\min(w,N_1)}}\left\{\binom{M}{w_1c}\right.\right.\\
&\quad\left.\left.\times\frac{\mathrm{coef}[F_{M-w_1c,w_1c,0}(x),x^{(w-w_1)c}]}{W_{w_1,w-w_1,0}}\right\}\right]\\
&\le2\omega^2N^2\max_{1\le w\le\omega N}\left[\binom{\omega N}{w}\max_{\substack{1\le w_1\le b_3\\ b_3=\min(w,N_1)}}\left\{\frac{\binom{M}{w_1c}}{W_{w_1,w-w_1,0}}\right.\right.\\
&\quad\left.\left.\times\mathrm{coef}[F_{M-w_1c,w_1c,0}(x),x^{(w-w_1)c}]\right\}\right].
\end{aligned}\tag{A· 13}$$

Substituting Eqs. (A· 12) and (A· 13) to Eq. (A· 11), we obtain

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\log\Pr[\mu(G_{\mathrm{LR}})\le\omega N]\\
&\le\lim_{N\to\infty}\frac{1}{N}\log\left[\left\{(N-2N_1-\omega N+1)\omega N+1\right\}\right.\\
&\times\max_{1\le w\le\omega N}\left\{\frac{\binom{\omega N}{w}}{W_{0,w,0}}\times\mathrm{coef}[F_{M,0,0}(x),x^{wc}]\right\}\\
&+2\omega^2N^2\max_{1\le w\le\omega N}\left[\binom{\omega N}{w}\max_{\substack{1\le w_1\le b_3\\ b_3=\min(w,N_1)}}\left\{\frac{\binom{M}{w_1c}}{W_{w_1,w-w_1,0}}\right.\right.\\
&\left.\left.\left.\times\mathrm{coef}[F_{M-w_1c,w_1c,0}(x),x^{(w-w_1)c}]\right\}\right]\right]\\
&\le\lim_{N\to\infty}\frac{1}{N}\log\left[\left\{(N-2N_1+\omega N+1)\omega N+1\right\}\right.\\
&\times\max_{1\le w\le\omega N}\left[\binom{\omega N}{w}\max_{\substack{0\le w_1\le b_3\\ b_3=\min(w,N_1)}}\left\{\frac{\binom{M}{w_1c}}{W_{w_1,w-w_1,0}}\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&\times \mathrm{coef}\left[F_{M-w_1c,w_1c,0}(x), x^{(w-w_1)c}\right]\Bigg\}\Bigg]\Bigg]\\
&\le \lim_{N\to\infty}\frac{1}{N}\log\Bigg[\Big\{(N-2N_1+\omega N+1)\omega N+1\Big\}\\
&\times \sup_{0<\alpha\le\omega}\Bigg[\binom{\omega N}{\alpha N}\sup_{\substack{0\le\alpha_1\le b_3\\ b_3=\min(\alpha,\frac{1}{d})}}\Bigg\{\frac{\binom{(1-R')N}{\alpha_1 Nc}}{W_{\alpha_1 N,(\alpha-\alpha_1)N,0}}\\
&\times \mathrm{coef}\left[F_{(1-R'-\alpha_1c)N,\alpha_1Nc,0}(x), x^{(\alpha-\alpha_1)Nc}\right]\Bigg\}\Bigg]\Bigg]\\
&= \lim_{N\to\infty}\frac{1}{N}\log\Big[(N-2N_1+\omega N+1)\omega N+1\Big]\\
&+ \lim_{N\to\infty}\frac{1}{N}\log\Bigg[\sup_{0<\alpha\le\omega}\Bigg[\binom{\omega N}{\alpha N}\sup_{\substack{0\le\alpha_1\le b_3\\ b_3=\min(\alpha,\frac{1}{d})}}\Bigg\{\frac{\binom{(1-R)N}{\alpha_1 Nc}}{W_{\alpha_1 N,(\alpha-\alpha_1)N,0}}\\
&\times \mathrm{coef}\left[F_{(1-R'-\alpha_1c)N,\alpha_1Nc,0}(x), x^{(\alpha-\alpha_1)Nc}\right]\Bigg\}\Bigg]\Bigg]. \qquad \text{(A· 14)}
\end{aligned}
$$

The limit of the first term of the r.h.s. in Eq. (A· 14) goes to zero as N tends to infinity.

$$
\lim_{N\to\infty}\frac{1}{N}\log\Big[(N-2N_1+\omega N+1)\omega N+1\Big]=0. \qquad \text{(A· 15)}
$$

The limit of the second term of this equation is given by

$$
\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\log\Bigg[\sup_{0<\alpha\le\omega}\Bigg[\binom{\omega N}{\alpha N}\sup_{\substack{0\le\alpha_1\le b_3\\ b_3=\min(\alpha,\frac{1}{d})}}\Bigg\{\frac{\binom{(1-R')N}{\alpha_1 Nc}}{W_{\alpha_1 N,(\alpha-\alpha_1)N,0}}\\
&\times \mathrm{coef}\left[F_{(1-R'-\alpha_1c)N,\alpha_1Nc,0}(x), x^{(\alpha-\alpha_1)Nc}\right]\Bigg\}\Bigg]\Bigg]\\
&= \sup_{0<\alpha\le\omega}\Bigg[\omega H\Big(\frac{\alpha}{\omega}\Big)+\lim_{N\to\infty}\frac{1}{N}\\
&\times\log\Bigg[\sup_{\substack{0\le\alpha_1\le b_3\\ b_3=\min(\alpha,\frac{1}{d})}}\Bigg\{\frac{\binom{(1-R')N}{\alpha_1 Nc}}{W_{\alpha_1 N,(\alpha-\alpha_1)N,0}}\\
&\times \mathrm{coef}\left[F_{(1-R'-\alpha_1c)N,\alpha_1Nc,0}(x), x^{(\alpha-\alpha_1)Nc}\right]\Bigg\}\Bigg]\Bigg]. \qquad \text{(A· 16)}
\end{aligned}
$$

The limit of the second term of the r.h.s. in Eq. (A· 16) is given by

$$
\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\log\Bigg[\sup_{\substack{0\le\alpha_1\le b_3\\ b_3=\min(\alpha,\frac{1}{d})}}\Bigg\{\frac{\binom{(1-R')N}{\alpha_1 Nc}}{W_{\alpha_1 N,(\alpha-\alpha_1)N,0}}\\
&\times \mathrm{coef}\left[F_{(1-R'-\alpha_1c)N,\alpha_1Nc,0}, x^{(\alpha-\alpha_1)Nc}\right]\Bigg\}\Bigg]\\
&= \sup_{\substack{0\le\alpha_1\le b_3\\ b_3=\min(\alpha,\frac{1}{d})}}\Bigg[(1-R')\Bigg\{H\Big(\frac{\alpha_1 c}{1-R'}\Big)-d_2H\Big(\frac{d(\alpha-\alpha_2)}{d_2}\Big)\\
&-H(d\alpha_1)\Bigg\}+\lim_{N\to\infty}\frac{1}{N}\log\Big\{\mathrm{coef}\big[F_{(1-R'-\alpha_1c)N,\alpha_1Nc,0}(x),\\
&x^{(\alpha-\alpha_1)Nc}\big]\Big\}\Bigg]. \qquad \text{(A· 17)}
\end{aligned}
$$

By using Lemma 2 [5], the limit of the second term of the r.h.s. in Eq. (A· 17) is given

$$
\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\log\Big\{\mathrm{coef}\big[F_{(1-R'-\alpha_1c)N,\alpha_1Nc,0}(x), x^{(\alpha-\alpha_1)Nc}\big]\Big\}\\
&= \log\frac{F_{1-R'-\alpha_1c,\alpha_1c,0}(x_0)}{x_0^{(\alpha-\alpha_1)c}}, \qquad \text{(A· 18)}
\end{aligned}
$$

where x_0 is a unique positive solution of

$$
\begin{aligned}
&\frac{(1-R'-\alpha_1c)d_2x\{(1+x)^{d_3}-1\}}{(1+x)^{d_2}-d_2x}\\
&+\frac{\alpha_1cd_2x(1+x)^{d_3}}{(1+x)^{d_2}-1}=(\alpha-\alpha_1)c. \qquad \text{(A· 19)}
\end{aligned}
$$

Substituting Eqs. (A· 15)–(A· 19) into Eq. (A· 14), we obtain Eq. (24).

Recall that α_1 obviously takes $0\le\alpha_1\le\frac{1}{d}$ and $2\alpha_1\le\alpha$ is satisfied since the α_1N erasures occurred at V_1, and then at least $(\alpha-\alpha_1)N$ erasures occurred at V_2 is needed to be a stopping set. □

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