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A Note on Performance of Generalized Tail Biting Trellis Codes

Shigeichi Hirasawa† and Masao Kasahara††

† Department of Industrial and Management Systems Engineering, School of Creative Science and Engineering, Waseda University, 3-4-1 Ohkubo Shinjuku, Tokyo, 169-8555 JAPAN.
E-mail: hira@waseda.jp

†† Faculty of Informatics, Osaka Gakuin University 2-36-1, Kishibe-minami, Suita, Osaka, 564-8511 JAPAN.
E-mail: kasahara@ogu.ac.jp

Abstract

Tail biting trellis codes and block concatenated codes are discussed from random coding arguments. Error exponents and decoding complexity for generalized tail biting (GTB) random trellis codes, and their relationships are derived, where the GTB trellis codes consist of full tail biting (FTB) trellis codes, partial tail biting (PTB) trellis codes and direct truncated (DT) trellis codes. We show that the PTB trellis codes at all rates except for low rates are superior among the GTB trellis codes, in a sense that they have smaller upper bound on the probability of decoding error for given decoding complexity. We then propose the generalized version of the block concatenated codes constructed by the GTB trellis inner codes, and derive error exponents and the decoding complexity for the proposed code. The results obtained show that the DT trellis inner codes are effective among the GTB trellis inner codes for constructing the generalized version of the concatenated codes to keep the same decoding complexity as the original concatenated codes. We also show that larger error exponents are obtained by the generalized version of concatenated codes, if the decoding complexity is allowed to be larger than that of the original concatenated code, although it is still in polynomial order.

1. Introduction

Generalized tail biting (GTB) trellis codes [7] are known to be one of the most powerful codes for converting trellis codes into block codes with no loss in rates. The GTB trellis codes consist of full tail biting (FTB) trellis codes, partial tail biting (PTB) trellis codes and direct truncated (DT) trellis coeds. Since the FTB trellis codes require an intolerable increase in decoding complexity, much efforts have been devoted to the studies on suboptimum decoding algorithms for the FTB trellis codes [1, 7] or efficient maximum likelihood decoding algorithms for the GTB trellis codes [9]. Unfortunately, however, the decoding complexity of the latter algorithms in worst cases is the same as that of the complete maximum likelihood decoding algorithm, although it is asymptotically the same as that of the Viterbi algorithm when the signal to noise ratio becomes large.

On the other hand, we assume the using of complete maximum likelihood decoding of the GTB trellis codes, since we are interested in random coding arguments. The q-ary GTB trellis code can be constructed as follows: Let the encoder be initialized by the last part $v'(\leq v)$ symbols of the information symbols of length u, where v is the constraint length of the trellis code, and ignore the output of the encoder corresponding to v' information symbols. Next input all u information symbols into the encoder yielding b channel symbols per information symbol, and output the codeword of length $N = ub$, where rate r is defined by $r = \frac{1}{b}\ln q$. We then have a (ub, u) block code over GF(q). Note that the case of $v' = v$ gives the FTB trellis code, and that of $v' = 0$, the DT trellis code. We have analyzed error exponents and the decoding complexity for the FTB random trellis code and the block concatenated [2] with the FTB trellis inner codes [4].

In this paper, we discuss the GTB trellis codes and the block concatenated codes with the GTB trellis inner codes. There is a possibility such that the probability of decoding error for the GTB trellis codes is smaller than

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that of ordinary block codes with the same decoding complexity, even if complete maximum likelihood decoding of the GTB trellis codes is performed. First, we derive the error exponents and the decoding complexity for the GTB trellis codes. We show that the PTB trellis codes at all rates except for low rates are superior among the GTB trellis codes, in a sense that they have smaller upper bound on the probability of decoding error for given decoding complexity. Second, the DT trellis coeds are compared with the terminated trellis codes. Next, we propose the generalized version of the block concatenated codes [6] constructed by the GTB trellis inner codes, and derive the error exponents and the decoding complexity for the proposed code. The results obtained show that the DT trellis inner codes are effective among the GTB trellis codes for constructing the generalized version of the concatenated codes to keep the same decoding complexity as the original concatenated codes [2]. Finally, we also show that by expense of the decoding complexity which is still in polynomial order, larger error exponents can be realized with PTB trellis inner cods for the generalized version of the concatenated codes.

Throughout this paper, assuming a discrete memoryless channel with capacity C, we discuss the lower bound on the reliability function (usually called the error exponent) and asymptotic decoding complexity measured by the computational work [8].

In Section 3, the error exponents and the decoding complexity for the GTB trellis codes are derived. The term $o(1)$s are disregarded, since we are interested in an asymptotic behavior. In Section 4, the block concatenated codes [2] and their generalized version [6] are discussed. The term $o(1)$s will appear, since the decoding complexity of the concatenated codes is given by a polynomial order in the code length. Section 5 describes conclusions and further works.

2. Preliminaries

2.1. Block codes

Let an (N, K) block code over GF(q) be a code of length N, number of information symbols K and rate R, where

$$R = \frac{K}{N} \ln q \quad (K \leq N). \quad \text{[nats/symbol]} \tag{1}$$

From random coding arguments for an ordinary block code, there exists a block code of length N and rate R for which the probability of decoding error $\Pr(\mathcal{E})$ and the decoding complexity G satisfy

$$\Pr(\mathcal{E}) \leq \exp[-NE(R)] \quad (0 \leq R < C) \tag{2}$$

and

$$G \sim \exp[NR], \tag{3}$$

where $E(\cdot)$ is (the lower bound on) the block code exponent [3], and the symbol "$\sim$" indicates asymptotic equality[1].

2.2. Trellis codes

Let a (u, v, b) trellis code over GF(q) be a code of branch length u, branch constraint length v, yielding b channel symbols per branch and rate r which satisfies

$$r = \frac{1}{b} \ln q. \quad \text{[nats/symbol]} \tag{4}$$

Hereafter, we denote $\frac{v}{u}$ by a parameter θ, i.e.,

$$\theta = \frac{v}{u} \quad (0 < \theta \leq 1). \tag{5}$$

Letting

$$N = ub \tag{6}$$

for direct truncated (DT) trellis codes and for terminated trellis codes, results derived are shown in Table 1, where $e(\cdot)$ is (the lower bound on) the trellis code exponent [3] (See [4] Appendix A).

[1] Strictly speaking, $G \sim N^2 \exp[NR]$ holds, since likelihood comparisons between two codewords require N logical operations and N shift operations, and the maximum number of comparisons of codewords is $\exp[NR]$. We have used $N^2 \exp[NR] = \exp NR[1 + o(1)]$, $o(1) = \frac{2 \ln N}{NR} \to 0$ as $N \to \infty$, where the term $o(1)$ is ignored in (3).

Table 1: Asymptotic results on error exponents and decoding complexity for block codes

Block code	Error exponent	Decoding complexity G	Upper bound on $\Pr(\cdot)$
Ordinary block code	$E(R)$	$\exp[NR]$	$G^{-\frac{E(R)}{R}}$
Terminated trellis code	$E(R)$ [3]	q^v	$G^{-\frac{1-\theta}{\theta}\frac{E(R)}{R}}$
GTB trellis code (Theorems 1 and 2)			
DT trellis code ($\theta' = 0$)	$E(r)$ [3]	q^v	$G^{-\frac{1}{\theta}\frac{E(r)}{r}}$
PTB trellis code ($0 < \theta' < \theta$)	$e_G(r)$	$q^{v+v'}$	$G^{-\frac{1}{\theta+\theta'}\frac{e_G(r)}{r}}$ [2]
FTB trellis code ($\theta' = \theta$)	$e_G(r)$	q^{2v}	$G^{-\frac{1}{2\theta}\frac{e_G(r)}{r}}$ †

Table 1 shows the error exponents and the decoding complexity for block codes. In this table, the rate R of the terminated trellis code is given by [3]:

$$R = \frac{(u-v)r}{u} = (1-\theta)r. \tag{7}$$

Note that the following equation holds between $E(R)$ and $e(r)$ [3]:

$$E(R) = \max_{0<\mu\leq 1}(1-\mu)e\left(\frac{R}{\mu}\right), \tag{8}$$

which is called the concatenation construction [3].

3. Generalized tail biting trellis codes

The GTB trellis code is introduced as a powerful converting method for maintaining a larger error exponent with no loss in rates, although the decoding complexity increases. The GTB trellis codes can be constructed as follows [7]: Suppose an encoder of a (u, v, b) trellis code. First, initialize the encoder by inputting the last v' information (branch) symbols of u information (branch) symbols, and ignore the output of the encoder. Next, input all u information symbols into the encoder, and output the codeword of length $N = ub$ in channel symbols. As the result, we have a (ub, u) block code of rate $r = \frac{1}{b}\ln q$ over $\mathrm{GF}(q)$ by the tail biting method. Hereafter we denote $\frac{v'}{u}$ by a parameter θ', i.e.,

$$\theta' = \frac{v'}{u}. \qquad (0 \leq \theta' \leq \theta \leq 1) \tag{9}$$

The GTB trellis codes are composed of:

(i) Direct truncated (DT) trellis codes for $v' = 0$ ($\theta' = 0$);

(ii) Partial tail biting (PTB) trellis codes for $0 < v' < v$ ($0 < \theta' < \theta \leq 1$); and

(iii) Full tail biting (FTB) trellis codes $v' = v$ ($\theta' = \theta \leq 1$).

3.1. Exponential error bounds for GTB trellis codes

Theorem 1 There exists a block code of length N and rate r obtained by a GTB random trellis code with $0 \leq \theta' \leq \theta \leq 1$ for which the probability of decoding error $\Pr(\mathcal{E})$ satisfies

$$\Pr(\mathcal{E}) \leq \exp[-Ne_G(r)] \quad (0 \leq r < C) \tag{10}$$

where

$$e_G(r) = \min\{\theta e(r), E[(1-\theta')r], E(\theta' r)\} \qquad (0 \leq \theta' \leq \theta \leq 1). \tag{11}$$

[2]Terms in an exponent part of G are taken to be minimum for given $0 \leq \theta' \leq \theta \leq 1$.

Proof: Let $\boldsymbol{w}$ be a message sequence of (branch) length u, where all messages are generated with the equal probability. Rewrite the sequence $\boldsymbol{w}$ as

$$\boldsymbol{w} = (\boldsymbol{w}_{u-v}, \boldsymbol{w}_v) \tag{12}$$

where $\boldsymbol{w}_{u-v}$ is the former part of $\boldsymbol{w}$ (length $u-v$), and $\boldsymbol{w}_v = (\boldsymbol{w}_{v'}, \boldsymbol{w}_{v-v'})$ the latter part of $\boldsymbol{w}$ (length v). First initialize the encoder by inputting $(\boldsymbol{w}_{v'}, 0^{v-v'})$, where 0^m is all 0 sequence of length m. Next input $\boldsymbol{w}$ into the encoder. Then output the coded sequence $\boldsymbol{x}$ of length[3] $N = ub$. Suppose the $q^{v'}$ Viterbi trellis diagrams, each of which starts at the state $s_i (i = 1, 2, \cdots, q^{v'})$ depending on $\boldsymbol{w}_{v'}$, and ends at the state $s_j \in \mathcal{S}_i$, where the number of the states s_j, i.e., $|\mathcal{S}_i|$, is $q^{v-v'}$ [7] (See Figure 1). The Viterbi decoder generates the maximum likelihood path $\boldsymbol{w}^{(i)}$ in the trellis diagram for starting at s_i and ending at $s_j \in \mathcal{S}_i$. Computing $\max_i \boldsymbol{w}^{(i)} = \hat{\boldsymbol{w}}$, the decoder outputs $\hat{\boldsymbol{w}}$. The decoding error occurs when $\{\boldsymbol{w} \neq \hat{\boldsymbol{w}}\}$. Without loss of generality, let the true path be $\boldsymbol{w} = 0^u$ which starts at s_1 (and ends at s_1). We then have three types of decoding error, i.e., $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ (See Figure 2).

The probability of decoding error $\Pr(\mathcal{E}_1)$ within the trellis diagram starting at s_1 (and ending at s_1) for a (u, v, b) random trellis code is given by [3]

$$\begin{aligned} \Pr(\mathcal{E}_1) &\leq K_1 N \exp[-vbE_0(\rho)] \quad (0 \leq \rho \leq 1) \\ &= \exp\{-N\theta[e(r) - o(1)]\} \end{aligned} \tag{13}$$

where an error event begins at any time and $o(1) = \frac{\ln K_1 N}{N\theta} \to 0$ as $N \to \infty$.

The probability of decoding error $\Pr(\mathcal{E}_2)$ within the trellis diagram starting at s_1 and ending at $s_j \in \mathcal{S}_1$ $(j \neq 1)$ is given by [3]

$$\begin{aligned} \Pr(\mathcal{E}_2) &\leq q^{(v-v')\rho} \exp\{-ubE_0(\rho)\} \\ &= \exp\{-N[E_0(\rho) - \rho(1-\theta')r]\} \\ &= \exp\{-NE[(1-\theta')r]\}, \end{aligned} \tag{14}$$

since the number of the possible adversaries is $q^{v-v'} - 1$.

While the probability of decoding error $\Pr(\mathcal{E}_3)$ within trellis diagrams starting at $s_i (i \neq 1, i = 2, 3, \cdots, q^{v'})$ and ending at $s_j \in \mathcal{S}_i$ is also given by

$$\begin{aligned} \Pr(\mathcal{E}_3) &\leq q^{v'\rho} \exp\{-ubE_0(\rho)\} \\ &= \exp\{-N[E_0(\rho) - \rho\theta' r]\} \\ &= \exp[-NE(\theta' r)], \end{aligned} \tag{15}$$

since the number of the possible adversaries is $q^{v'} - 1$.

From (13),(14) and (15), the probability of over-all decoding error $\Pr(\mathcal{E})$ is bounded by the union bound:

$$\begin{aligned} \Pr(\mathcal{E}) &\leq \Pr(\mathcal{E}_1) + \Pr(\mathcal{E}_2) + \Pr(\mathcal{E}_3) \\ &= \exp\{-N[e_G(r) - o(1)]\}, \end{aligned} \tag{16}$$

where $e_G(r)$ is given by (11) and $o(1) = \frac{\ln K_1 N}{N\theta} + \frac{\ln 3}{N} \to 0$ as $N \to \infty$. Disregard of the term $o(1)$ in (16) gives (11). $\square$

Note that in the proof of Theorem 1, not only the trellis code construction but the block code construction appear as shown in Figure 2.

In our previous paper [4], we have discussed the FTB trellis codes, where we restricted ourselves to be $0 < \theta \leq \frac{1}{2}$, since larger error exponents which are dominated by $e(\cdot)$ are obtained and the decoding complexity for the block concatenated code with the FTB trellis inner codes is remained to that of the original concatenated codes.

Collorary 1 (FTB trellis codes [4]) The probability of decoding error $\Pr(\mathcal{E})$ for the FTB trellis codes with $0 < \theta \leq \frac{1}{2}$ satisfies

$$\Pr(\mathcal{E}) \leq \exp[-N\theta e(r)] \quad (0 \leq r < C) \tag{17}$$

[3]Note that GTB random trellis coding requires every channel symbol on every branch be chosen independently at random with the probability $\boldsymbol{p}$ which maximizes $E_0(\rho, \boldsymbol{p})$ on nonpathological channels [3].

Starting states	Ending states	Starting states	Ending states
$s_1 = (0,0)$,	$S_1 = \{(0,0)\}$	$s_1 = (0,0)$,	$S_1 = \{(0,0),(0,1)\}$
$s_2 = (0,1)$,	$S_2 = \{(0,1)\}$		
$s_3 = (1,0)$,	$S_3 = \{(1,0)\}$	$s_3 = (1,0)$,	$S_3 = \{(1,0),(1,1)\}$
$s_4 = (1,1)$,	$S_4 = \{(1,1)\}$		

(a) FTB trellis code for $v = v' = 2$ (b) PTB trellis code for $v = 2$, and $v' = 1$

Figure 1: Examples of FTB and PTB trellis codes

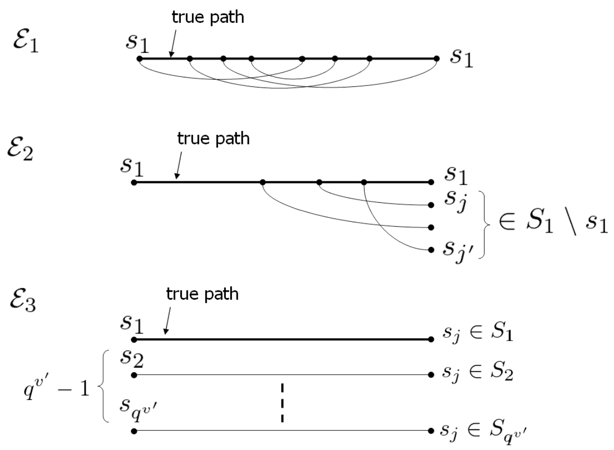

Figure 2: Three types of error events

and the decoding complexity G for the FTB trellis codes is given by

$$G \sim q^{2v} = \exp[2N\theta r]. \tag{18}$$

□

Comparison between the DT trellis codes and terminated trellis codes leads the following corollary.

Collorary 2 The DT trellis codes have a smaller upper bound on the probability of decoding error than both the terminated trellis codes and the ordinary block codes at the same decoding complexity.
Proof: See Appendix A. □

Example 1 On a very noisy channel[4], the exponents for the GTB trellis codes are depicted in Figure 3(a), (b) and Figure 4. We see that:

(a) The largest exponent is obtained at $\theta' = 0.5$ for $\frac{1}{2} < \theta \leq 1$, since $E[(1-\theta')r] = E(\theta' r)$ holds, hence the PTB trellis codes are the best among GTB trellis codes from the view-point of error exponents.

(b) While the largest exponent is obtained at $\theta = \theta'$ for $0 < \theta \leq \frac{1}{2}$, hence the FTB trellis codes are the best among the GTB trellis codes.

□

[4]On a very noisy channel, the upper bound and the lower bound to the error exponent are approximately the same for all rates, hence it is called the true error exponent. The error exponents of orthogonal codes over the unlimited bandwidth white Gaussian channel coincide with those of codes over a very noisy channel [2].

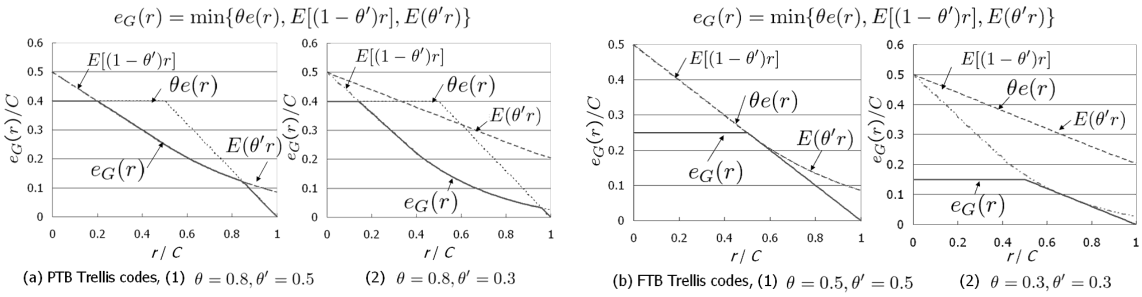

Figure 3: Example of construction of error exponents for GTB trellis codes over a very noisy channel

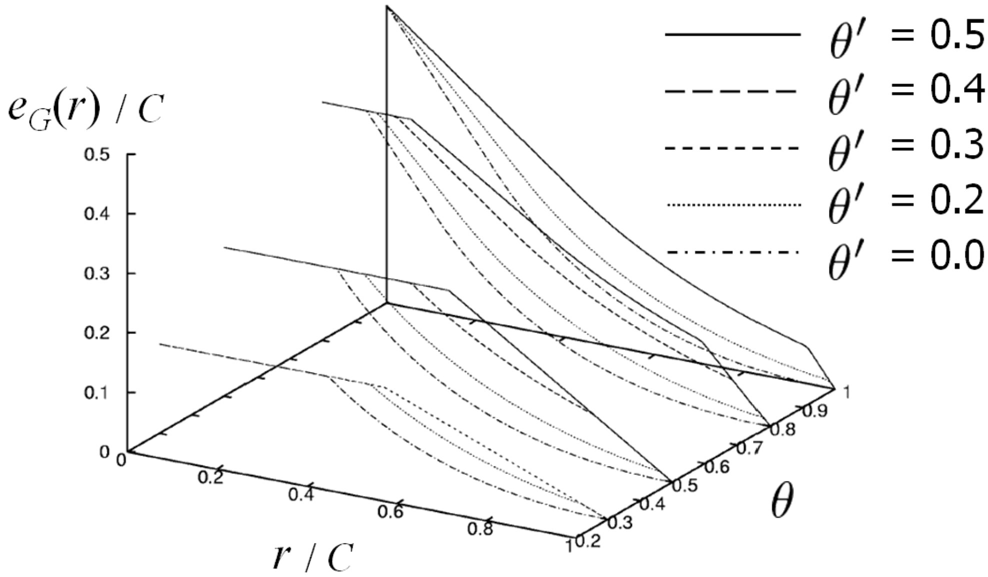

Figure 4: Exponents $g_G(r)$ of G for a very noisy channel.

3.2. Decoding complexity for GTB trellis codes

The maximum likelihood decoder for the Viterbi algorithm requires $u^2 q^{v+1}$ comparisons (See derivations in [4] Appendix A) for each trellis diagram and performs them in parallel for $q^{v'}$ trellis diagrams for the GTB trellis codes. We then have Theorem 2, where $u^2 q^{v+v'+1} = u^2 q q^{v+v'} = \exp\{Nr[\theta + \theta' + o(1)]\}$ ($o(1) = (\frac{2\ln u + \ln q}{vbr} \to 0$ as $v \to \infty$) and $q = \exp[br]$ are used.

Theorem 2 The decoding complexity G of the GTB trellis code is given by

$$G \sim q^{v+v'} = \exp[N(\theta + \theta')r] \quad (0 \le \theta' \le \theta \le 1) \tag{19}$$

□

The results derived in Theorem 1 and Theorem 2 are also shown in Table 1.

3.3. Upper bound on probability of decoding error for same decoding complexity

Next, we evaluate the probability of decoding error $\Pr(\mathcal{E})$ by taking into account the decoding complexity G so that coding methods can be clearly compared [3].

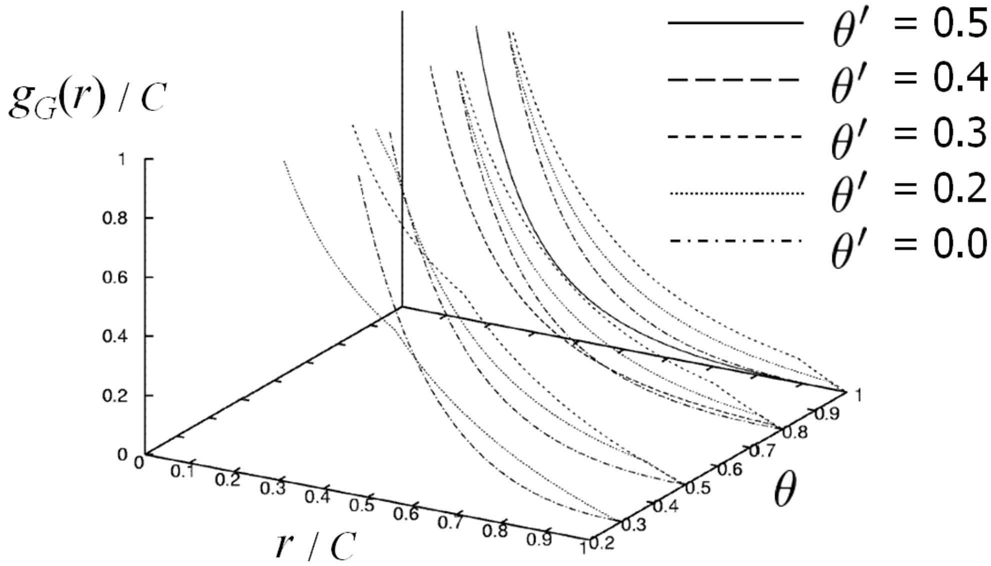

Figure 5: Error exponents $e_G(r)$ of GTB trellis codes for a very noisy channel.

Let us assume that the code length N and rate $R = r$ are the same for all conversion methods. To rewrite $\Pr(\mathcal{E})$ in terms of G for the ordinary block codes, we have $G \sim \exp[NR]$ from (3), i.e.,

$$N \sim \frac{1}{R} \ln G. \tag{20}$$

We then have [3]

$$\Pr(\mathcal{E}) \leq G^{-\frac{E(R)}{R}} \tag{21}$$

Since (19) holds for the GTB trellis code, we have the following corollary.

Collorary 3 For the GTB trellis code, we have

$$\Pr(\mathcal{E}) \leq G^{-g_G(r)}, \tag{22}$$

where

$$g_G(r) = \frac{1}{(\theta + \theta')r} \min\{\theta e(r), E[(1-\theta')r], E(\theta' r)\}, \tag{23}$$

and the term $g_G(r)$ in an exponent part of G is taken to be minimum for $0 \leq \theta' \leq \theta \leq 1$.
Proof: See Appendix B. □

A similar derivation gives the evaluations for the DT trellis code and for terminated trellis code as shown in Table 1, where $q^v = \exp[vbr] = \exp[N\theta r]$ holds (See [4] Appendix A).

Example 2 On a very noisy channel, the exponents $g_G(r)$ of G in (23) for the GTB trellis code are shown in Figure 5. We see that:

(a) For $\frac{1}{2} < \theta \leq 1$, $g_G(r)$ is the largest at $\theta' = 0.5$, hence the PTB trellis codes are the best among the GTB trellis codes at all rates except for low rates. The DT and FTB trellis codes have smaller values of $g_G(r)$ than the PTB trellis codes.

(b) While for $0 < \theta \leq \frac{1}{2}$, $g_G(r)$ is the largest at $\theta = \theta'$, hence the FTB trellis codes are the best among the GBT codes, and the DT trellis codes have smaller values of $g_G(r)$ than the FTB trellis codes.

□

4. Generalized version of concatenated codes with GTB trellis inner codes

In our previous paper [4], we have discussed the concatenated codes $\mathcal{C}_T$ with the FTB trellis inner codes. By using the FTB trellis coeds as the inner codes, we can attain a larger error exponent without increasing the decoding complexity of the concatenated codes. The result obtained is summarized in Lemma 1.

Lemma 1 [4] There exits a block concatenated code $\mathcal{C}_T$ of length N_0 and rate R_0 for which the probability of decoding error $\Pr(\mathcal{E})$ satisfies

$$\Pr(\mathcal{E}) \leq \exp[-N_0\theta e_C(R_0)] \quad \left(0 < \theta \leq \frac{1}{2}, 0 \leq R_0 < C\right) \tag{24}$$

where

$$e_C(R_0) = \max_{0<r<C}\left(1 - \frac{R_0}{r}\right) e(r) \tag{25}$$

and the over-all decoding complexity $G_0(N_0)$ for a block concatenated code $\mathcal{C}_T$ of length N_0 is give by

$$G_0(N_0) = O(N_0^2 \log^2 N_0) \quad \left(0 < \theta \leq \frac{1}{2}\right) \tag{26}$$

□

Note that $e_C(R_0) = E(R_0)$ holds for $0 \leq R_0 < C$, hence larger error exponents are successfully obtained by using the FTB trellis code as the inner code. Eq. (26) holds as far as $\theta + \theta' \leq 1$.

For the case of the generalized version of concatenated codes, called Codes $\mathcal{C}^{(J)}$ [6], the situation will be considerably changed. Since Codes $\mathcal{C}^{(J)}$ requires J (n,k) RS codes as the outer code, $n = q^{\frac{u}{J}}$ must be satisfied [6]. This implies that we must choose $\theta + \theta' = \frac{1}{J}$, since the decoding complexity of the inner codes is given by $O(n^{1+J(\theta+\theta')}\log^2 n)$, hence a small θ results in a small error exponent from (10) and (11). Consequently, we may choose the DT trellis codes as the inner codes of Codes $\mathcal{C}^{(J)}$.

We have used the terminated trellis codes as the inner codes for Codes $\mathcal{C}^{(J)}$ [6] to reduce the over-all decoding complexity. The following theorem states that the performance of the DT trellis codes with $\theta' = 0$ and $\theta = \frac{1}{J}$ is the same as that of the terminated trellis codes.

Theorem 3 The probability of decoding error $\Pr(\mathcal{E})$ for Codes $\mathcal{C}^{(J)}$ with the DT trellis inner codes is the same as that with the terminated trellis inner codes [6] where $\Pr(\mathcal{E})$ is given by [6]

$$\Pr(\mathcal{E}) \leq \exp[-N_0 E_C^{(J)}(R_0)] \quad (0 \leq R_0 \leq C), \tag{27}$$

and

$$E_C^{(J)}(R_0) = \max_R\left(1 - \frac{R_0}{R}\right)\frac{J}{\sum_{j=1}^{J}\left[\frac{E(R)}{E(\frac{jR}{J})}\right]}E(R). \tag{28}$$

The decoding complexity for Codes $\mathcal{C}^{(J)}$ with the both inner codes is given by $O(N_0^2 \log^2 N_0)$, where N_0 is the over-all length of Codes $\mathcal{C}^{(J)}$. □

The result obtained in Theorem 3 shows that the DT trellis inner codes are effective among the GTB trellis codes for constructing Codes $\mathcal{C}^{(J)}$ to keep the same decoding complexity as the original concatenated codes.

Since we cannot choose $\theta = \frac{1}{J}$ for the FTB trellis codes, we have the following theorem by expense of increasing the decoding complexity to Codes $\mathcal{C}^{(J)}$.

Theorem 4 The probability of decoding error $\Pr(\mathcal{E})$ for Codes $\mathcal{C}^{(J)}$ with the GTB trellis codes is given by

$$\Pr(\mathcal{E}) \leq \exp[-N_0 e_C^{(J)}(R_0)] \tag{29}$$

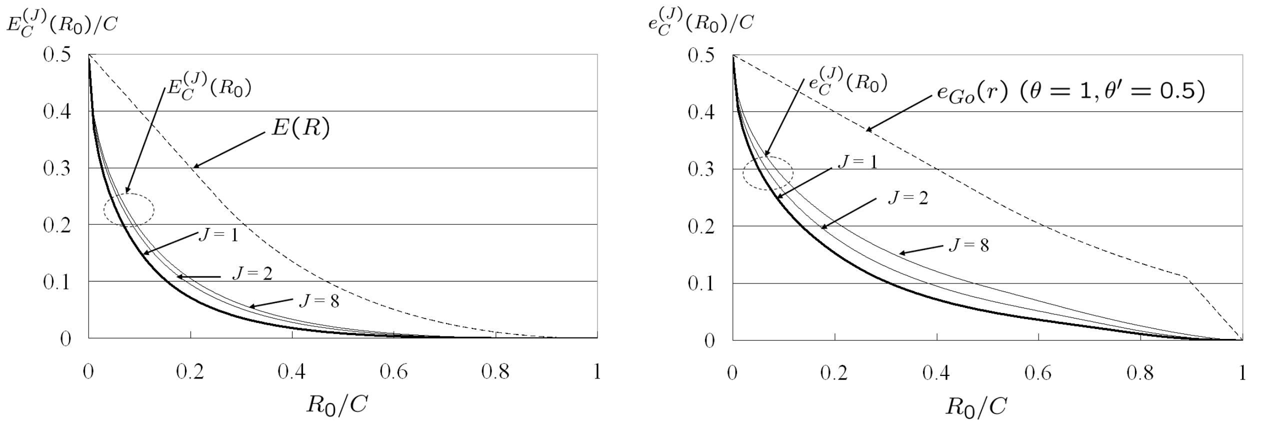

Figure 6: (a) Code $C^{(J)}$ with DT trellis inner codes over a very noisy channel

Figure 6: (b) Code $C^{(J)}$ with PTB trellis inner codes over a very noisy channel

Figure 6: Error exponent $e_{G_0}(r)$ for GTB trellis codes, (a) those $e_C^{(J)}(R_0)$ and (b) those $E_C^{(J)}(R_0)$ for Codes $\mathcal{C}^{(J)}$

where

$$e_C^{(J)}(R_0) = \max_r \Big(1 - \frac{R_0}{r}\Big) \frac{J}{\sum_{j=1}^{J} \Big[\frac{e_G(r)}{e_G(\frac{jr}{J})}\Big]} e_G(r). \tag{30}$$

While the decoding complexity $G(N_0)$ is given by

$$G(N_0) = \begin{cases} O(N_0^2 \log^2 N_0), & J = 1; \\ O(N_0^{1+J(\theta+\theta')} \log^{1-J(\theta+\theta')} N_0), & J \geq 2, \end{cases} \tag{31}$$

Proof: See Appendix C.

The results obtained in this section are summarized in Table 2.

Example 3 On a very noisy channel, the exponents for Codes $\mathcal{C}^{(J)}$ are shown in Figure 6, where the exponents are chosen to be the largest for θ and θ'. Rewrite $e_G(r)$ in (11) to $e_G(r, \theta, \theta')$ as a function of θ and θ'. Then choose θ_0 and θ'_0 as

$$(\theta_0, \theta'_0) = \arg \max_{0 \leq \theta' \leq \theta \leq 1} e_G(r, \theta, \theta'), \tag{32}$$

we then have

$$e_{G0}(r) = e_G(r, \theta_0, \theta'_0), \tag{33}$$

where (θ, θ') are depicted in Figure 7.

Next, substitution $e_{G_0}(r)$ of (31) into $e_G(r)$'s in (30) gives $e_C^{(J)}(R_0)$ which are illustrated in Figure 6 together with $E_C^{(J)}(R_0)$. □

5. Concluding remarks

We have derived the error exponents and the decoding complexity for block codes converted from the GTB trellis codes. We have shown that the PTB trellis codes at all rates except for low rates are superior among the GTB trellis codes, in a sense that they have the smallest upper bound on the probability of decoding error for given decoding complexity. This result suggests us that we can attain high performance by the PTB trellis codes with a careful choice of the parameter θ' for given θ. It has been also clarified that the DT trellis inner codes are

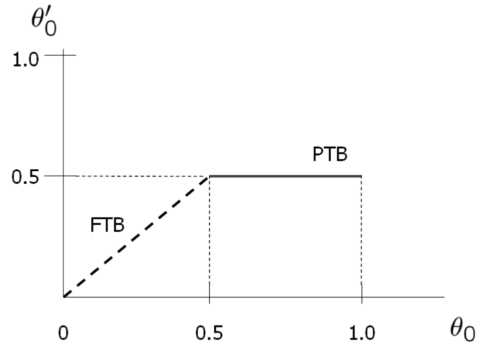

Figure 7: Optimum parameters θ_0 and θ_0'

Table 2: Error exponents and decoding complexity for Codes $\mathcal{C}^{(J)}$

Code $\mathcal{C}^{(J)}$	inner codes	error exponents	decoding complexity
Original concatenated codes [2] ($J=1$)	ordinary block codes	$E_C(R_0)=\max_R(1-\frac{R_0}{R})E(R)$	$O(N_0^2 \log^2 N_0)$
$J \geq 2$	teminated trellis codes [6] DT trellis codes [5]	$E_C^{(J)}(R_0)=\max_R\left(1-\frac{R_0}{R}\right)$ $\times \frac{J}{\sum_{j=1}^{J}\left[\frac{E(R)}{E(\frac{jR}{J})}\right]} E(R)$	$O(N_0^2 \log^2 N_0)$
$J=1$	FTB trellis codes [4] $(0 \leq \theta' \leq \theta \leq 1)$	$\theta e_C(R_0)=\max(1-\frac{R_0}{r})e(r)$	$O(N_0^2 \log^2 N_0)$
$J \geq 1$	GTB trellis codes [Theorem 4]	$e_C^{(J)}(R_0)=\max_r\left(1-\frac{R_0}{r}\right)$ $\times \frac{J}{\sum_{j=1}^{J}\left[\frac{e_G(r)}{e_G(\frac{jr}{J})}\right]} e_G(r)$	$O(N_0^2 \log^2 N_0),\ J=1$ $O(N_0^{1+J(\theta+\theta')} \log^{1-J(\theta+\theta')} N_0),\ J \geq 2$

effective among the GTB trellis codes for constructing the generalized version of concatenated codes, Codes $\mathcal{C}^{(J)}$, to keep the same decoding complexity as the original concatenated codes.

If we can allow increasing the decoding complexity, larger exponents are obtained by Codes $\mathcal{C}^{(J)}$ with the GTB trellis inner codes.

A detailed analysis on upper bounds on the probability of decoding error for the GTB trellis codes with different parameters θ and θ' at the same decoding complexity will be in further investigation. Although the random coding arguments suggest some useful aspects to construct the code, we should note to make them applicable to a practical code, which is also a future work.

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A. Proof of Corollary 2

From Table 1, we see that

$$\frac{E(R)}{R} \leq \frac{(1-\theta)E(R)}{\theta R} \leq \frac{E(r)}{\theta r} \quad \left(0 < \theta \leq \frac{1}{2}\right) \tag{34}$$

and

$$\frac{(1-\theta)E(R)}{\theta R} \leq \frac{E(R)}{R} \leq \frac{E(r)}{\theta r} \quad \left(\frac{1}{2} < \theta \leq 1\right), \tag{35}$$

completing the proof.

B. Proof of Corollary 3

Substitution of (19) into (10) after a little manipulation gives

$$\begin{aligned} \Pr(\mathcal{E}) &\leq \exp[-Ne_G(r)] \\ &= G^{-\frac{Ne_G(r)}{N(\theta+\theta')r}}. \end{aligned} \tag{36}$$

C. Proof of Theorem 4

Since the error exponents of the GTB trellis codes are given by (11), those $e_C^{(J)}(R_0)$ of Codes $\mathcal{C}^{(J)}$ $(J \geq 1)$ with the GTB trellis inner codes can be easily derived as (30) similar to [6] by using the construction of subcodes of GTB trellis codes. While the over-all decoding complexity $G_0(N_0)$ is given by

$$G_O(N_0) = \max\big[G_I(n,N), G_O(n)\big] = \begin{cases} O(N_0^2 \log^2 N_0), & J = 1, \\ O(N_0^{1+J(\theta+\theta')} \log^{1-J(\theta+\theta')} N_b), & J \geq 2, \end{cases} \tag{37}$$

since the decoding complexity for the GTB trellis inner codes $G_I(n,N)$ and that of the RS outer codes $G_O(n)$ are given by

$$G_I(n,N) = O(nN^2 q^{(v+v')}), \qquad J \geq 1, \tag{38}$$

and

$$G_O(n) = O(n^2 \log^4 n), \qquad J \geq 1. \tag{39}$$

Note that $G_0(N_0)$ is dominated by $G_O(n)$ for $J = 1$, and by $G_I(n,N)$ for $J \geq 2$, where we have used

$$n = O(N_0/\log N_0). \tag{40}$$

References

[1] R. V. Cox and C-E W. Sundberg, "An efficient adaptive circular Viterbi algorithm for decoding generalized tailbiting convolutional codes," *IEEE Trans. Vehicular Technology*, vol.43, no.1, pp.57-68, Feb. 1994.

[2] G. D. Forney, Jr., *Concatenated Codes*, Cambridge, MA. MIT, 1966.

[3] G. D. Forney, Jr., "Convolutional codes II. Maximum-likelihood decoding," *Inform. Contr.* vol.25, pp.222-266, 1976.

[4] S. Hirasawa and M. Kasahara, "Exponential error bounds and decoding complexity for block concatenated codes with tail biting trellis inner codes," *J. Discrete Mathematical Sciences and Cryptography*, vol.9, no.2, pp.307-320, Aug. 2006.

[5] S. Hirasawa and M. Kasahara, "Exponential error bounds and decoding complexity for generalized tail biting trellis codes," *Proceeding of The 31-st Symposium on Information Theory and its Application* (SITA 2008), pp. 918-923, Kinugawa, Tochigi, Japan, Oct. 7-10, 2008.

[6] S. Hirasawa, M. Kasahara, Y. Sugiyama, and T. Namekawa, "Certain generalizations of concatenated codes-Exponential error bounds and decoding complexity," *IEEE Trans. Inform. Theory*, vol.IT-26, no.5, pp.527-534, Sept.1980.

[7] H. H. Ma, and J. K. Wolf, "On tail biting convolutional codes," *IEEE Trans. Communications*, vol.COM-34, no.2, pp.104-111, Feb.1986.

[8] J. E. Savage, “The complexity of decoders-Part II: Computational work and decoding time,” *IEEE Trans. Inform. Theory*, vol.IT-17, no.1, pp.77-85, Jan.1971.

[9] Q. Wang and V. K. Bhargava, “An efficient maximum-likelihood decoding for generalized tail biting convolutional codes including quasicyclic codes,” *IEEE Trans. Communications*, vol.Com-37, no.8, pp.875-879, Aug. 1989.