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A Note on Performance of Generalized Tail Biting Trellis Codes

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1. Introduction

Coding Theorem … random coding arguments

- existence of a code
- essential behavior of the code
- quantitative evaluation

$\Pr(\mathcal{E})$: probability of decoding error

R : rate

G : decoding complexity

Coding theorem aspects → practical coding problem

- **Generalized tail biting (GTB) trellis codes**

  c.f. Ordinary block codes/Terminated trellis codes

  ･Direct truncated (DT) trellis codes [3]

  ･Partial tail biting (PTB) trellis codes

  ･Full tail biting (FTB) trellis codes [4,5]

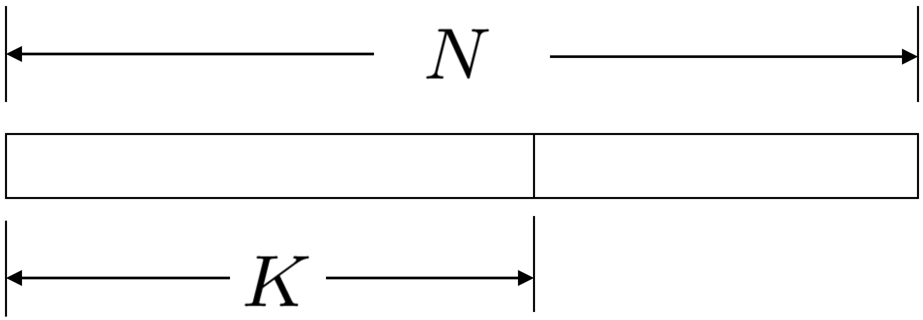
- **Generalized version of concatenated codes with generalized tail biting trellis inner codes --- Codes $C^{(J)}$ [6]**

  - inner codes --- GTB trellis codes
  - outer codes --- J Reed Solomon (RS) codes

2. Preliminaries

2.1 Block codes

(N, K) block code



N: code length

K: number of information symbols

R: rate

$$R = \frac{K}{N} \ln q \quad [\text{nats/symbol}] \tag{1}$$

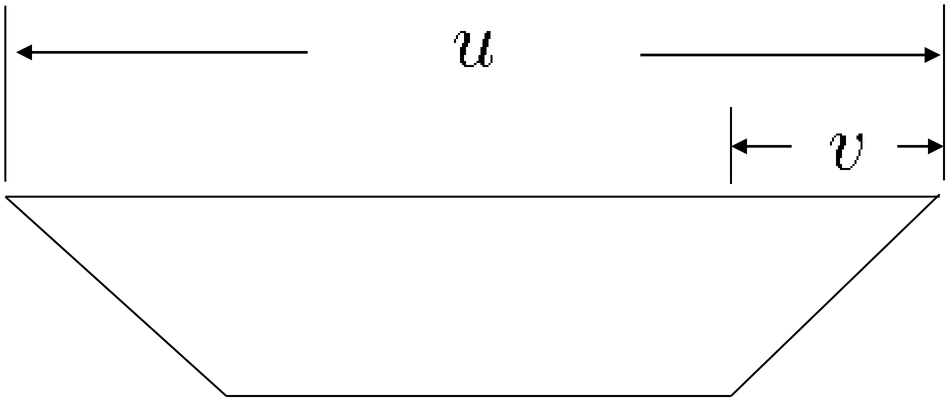
$$\Pr(\mathcal{E}) \le \exp[-NE(R)] \;\; (0 \le R < C) \tag{2}$$

$$G \sim \exp[NR] \tag{3}$$

$E(R)$: block code exponent

2.2 Trellis codes

(u, v, b) trellis code



u: branch length

v: branch constraint length

b: number of channel symbols / branch

r: rate

$$r = \frac{1}{b} \ln q \quad \text{[nats/symbol]} \tag{4}$$

$$\Pr(\mathcal{E}) \le \exp[-vbe(r)] \ (0 \le r < C)$$

$$G \sim q^v$$

$e(r)$: trellis code exponent

parameter $\theta = \dfrac{v}{u} \quad (0 < \theta \le 1)$

Block codes converted from trellis codes

(N, K) terminated trellis code [3]

$$N = ub \qquad (6)$$

$$K = u$$

$$R = (1-\theta)r \qquad (7)$$

$$\Pr(\mathcal{E}) \le \exp[-NE(R)]$$

$$E(R) = \max_{0<\mu\le 1} (1-\mu)e(R/\mu) \qquad (8)$$

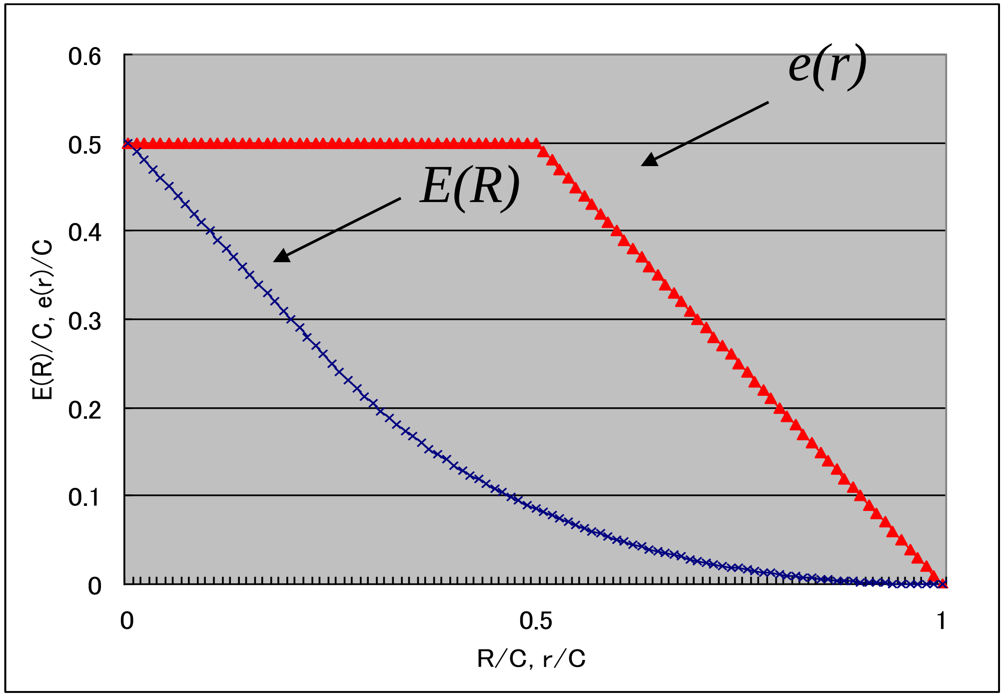
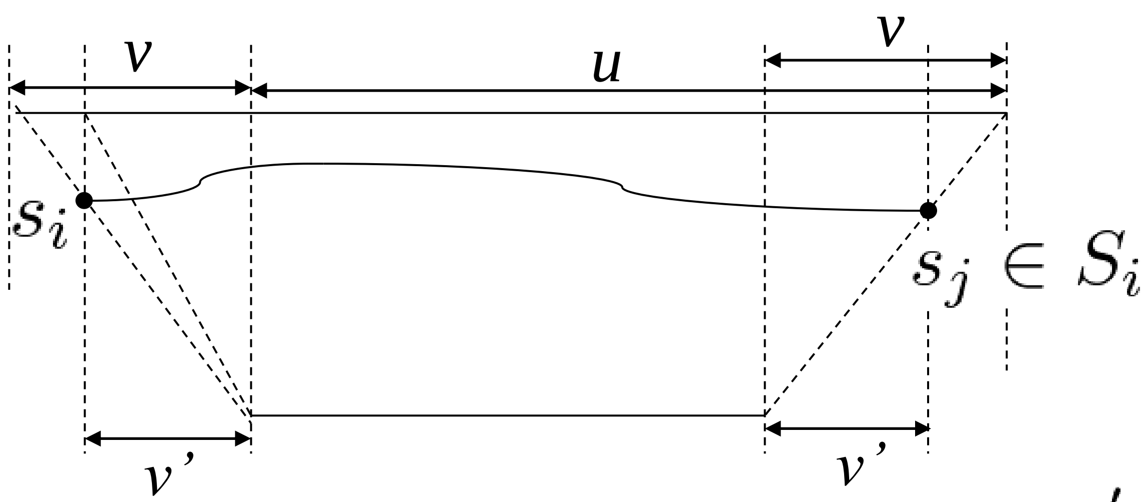


Fig.0: Block code exponent and trellis code exponent for very noisy channel

3. Generalized tail biting trellis codes

(N, K) GTB trellis code



$$\theta = \frac{v}{u}$$

$$\theta' = \frac{v'}{u}$$

$$|S_i| = q^{v-v'}$$

$$N = ub, \quad K = u, \quad r = \frac{1}{b} \ln q \qquad \text{parameter} \quad \theta' = \frac{v'}{u} \tag{9}$$

(i) DT (direct truncated) [3]: $v' = 0 \quad (\theta' = 0)$

$$\Pr(\mathcal{E}) \leq \exp[-NE(r)]$$

(ii) PTB (partial tail biting): $0 < v' < v \quad (0 < \theta' < \theta \leq 1)$

(iii) FTB (full tail biting) [4, 5]: $v' = v \quad (\theta' = \theta)$

$$\Pr(\mathcal{E}) \leq \exp[-N\theta e(r)] \;\; (0 < \theta \leq 1/2)$$

Starting states	Ending states	Starting states	Ending states
$s_1 = (0,0),$	$S_1 = \{(0,0)\}$	$s_1 = (0,0),$	$S_1 = \{(0,0),(0,1)\}$
$s_2 = (0,1),$	$S_2 = \{(0,1)\}$		
$s_3 = (1,0),$	$S_3 = \{(1,0)\}$	$s_3 = (1,0),$	$S_3 = \{(1,0),(1,1)\}$
$s_4 = (1,1),$	$S_4 = \{(1,1)\}$		

(a) FTB trellis code for $v = v' = 2$

(b) PTB trellis code for $v = 2$, and $v' = 1$

Figure1: Examples of FTB and PTB trellis codes

3.1 Exponential error bounds for GTB Trellis codes

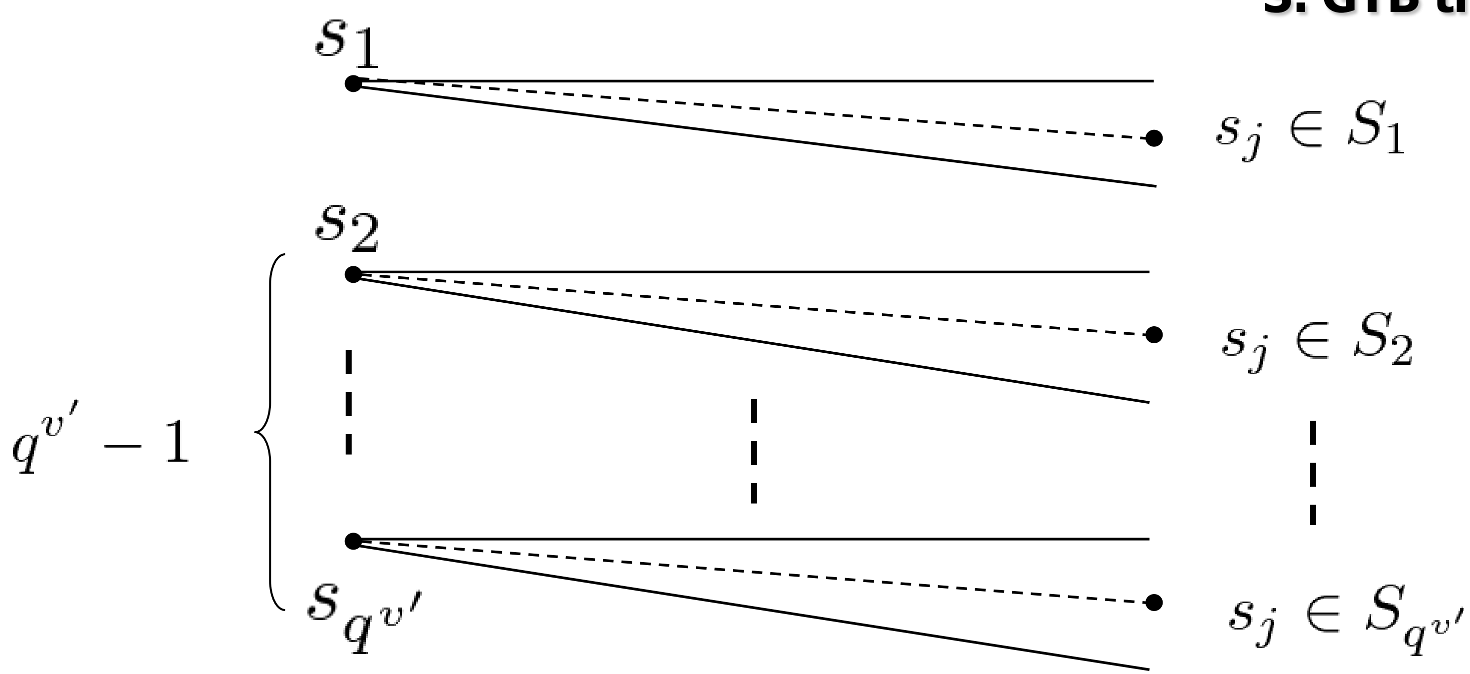
[Theorem 1] GTB trellis code

For $0 \leq \theta' \leq \theta \leq 1$

$$\Pr(\mathcal{E}) \leq \exp[-Ne_G(r)] \quad (0 \leq r < C) \qquad (10)$$

where

$$e_G(r) = \min\{\theta e(r), E[(1-\theta')r], E(\theta' r)\} \qquad (11)$$



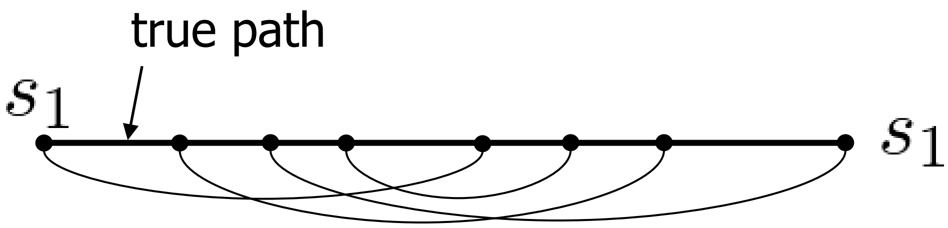
true path : 0^u

$\mathcal{E}_1$: starts at $s_1 = 0^v$ and ends at $s_1 = 0^v$

$\mathcal{E}_2$: starts at $s_1 = 0^v$ and ends at $s_j \in S_1 \setminus s_1$

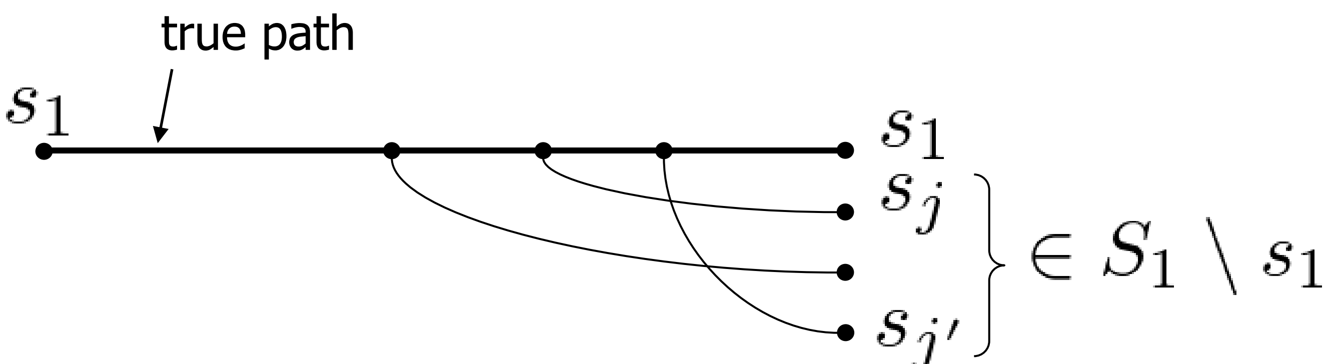
$\mathcal{E}_3$: starts at $s_i = 0^v \quad (i \neq 1)$ and ends at $s_j \in S_1 \setminus s_1$

$\mathcal{E}_1$

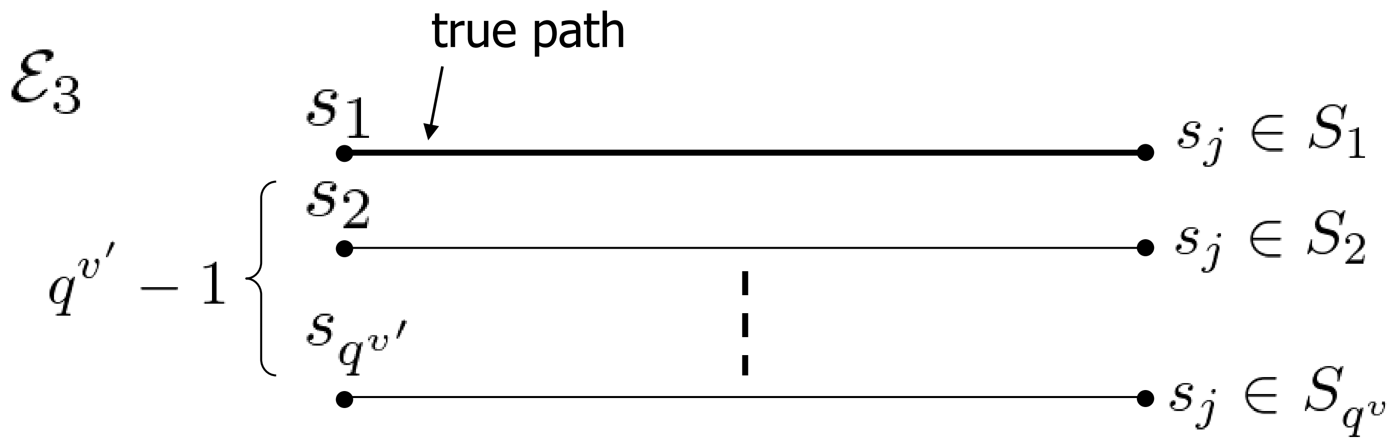


$$\begin{aligned}\Pr(\mathcal{E}_1) &\leq K_1 N \exp[-vbE_0(\rho)] \quad (0 \leq \rho \leq 1) \\ &= \exp\{-N\theta[e(r) - o(1)]\} \qquad (13)\end{aligned}$$

$\mathcal{E}_2$



$$\begin{aligned}\Pr(\mathcal{E}_2) &\leq q^{(v-v')\rho} \exp\{-ubE_0(\rho)\} \\ &= \exp\{-N[E_0(\rho) - \rho(1-\theta')r]\} \\ &= \exp\{-NE[(1-\theta')r]\}, \qquad (14)\end{aligned}$$

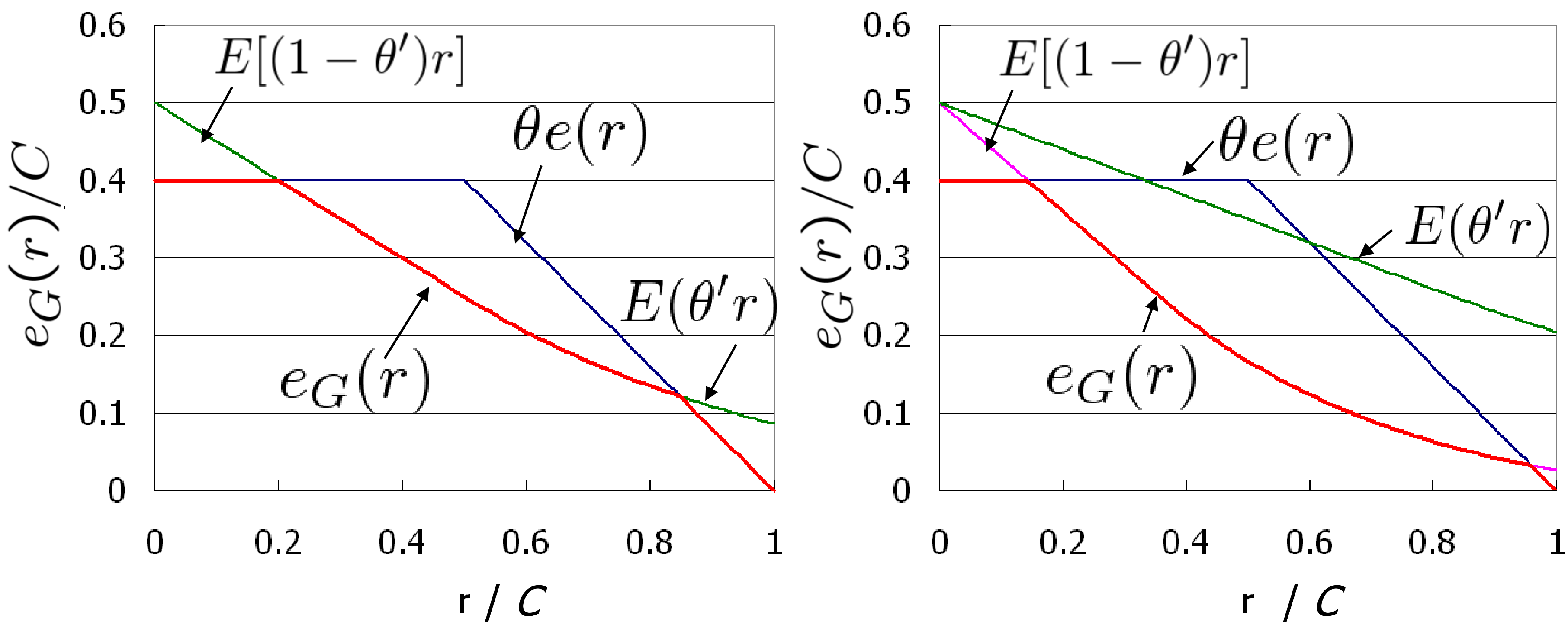


$$\begin{aligned} \Pr(\mathcal{E}_3) &\leq q^{v'\rho} \exp\{-ubE_0(\rho)\} \\ &= \exp\{-N[E_0(\rho) - \rho\theta' r]\} \\ &= \exp[-NE(\theta' r)], \end{aligned} \quad (15)$$

$$\begin{aligned} \Pr(\mathcal{E}) &\leq \Pr(\mathcal{E}_1) + \Pr(\mathcal{E}_2) + \Pr(\mathcal{E}_3) \\ &\leq 3\exp[-N\min\{\theta e(r), E[(1-\theta')r], E(\theta' r)\}] \end{aligned} \quad (16)$$

[Example 1] GTB trellis codes for Very Noisy Channel

$$e_G(r) = \min\{\theta e(r), E[(1-\theta')r], E(\theta' r)\}$$

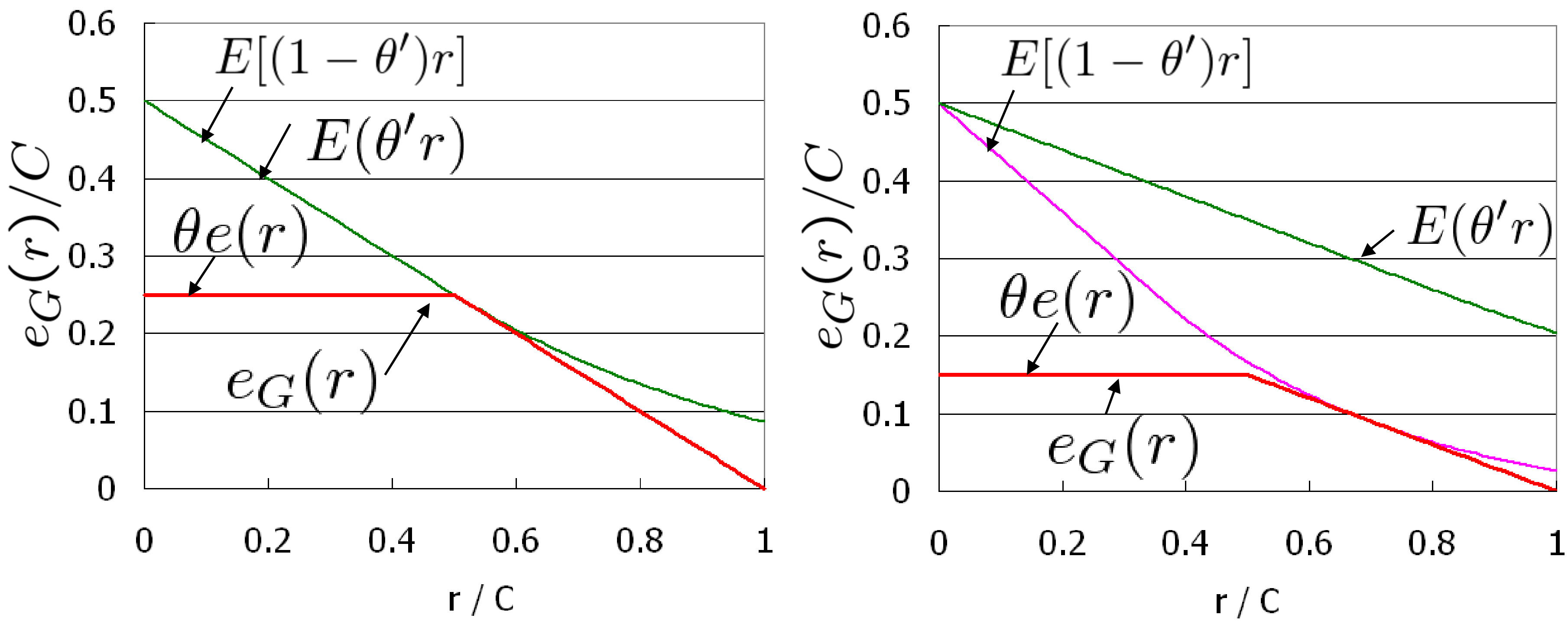


(a) PTB Trellis codes, (1) $\theta = 0.8, \theta' = 0.5$ (2) $\theta = 0.8, \theta' = 0.3$

$$E[(1-\theta)r] \geq \theta e(r) \quad (0 < r \leq C)$$ [3]

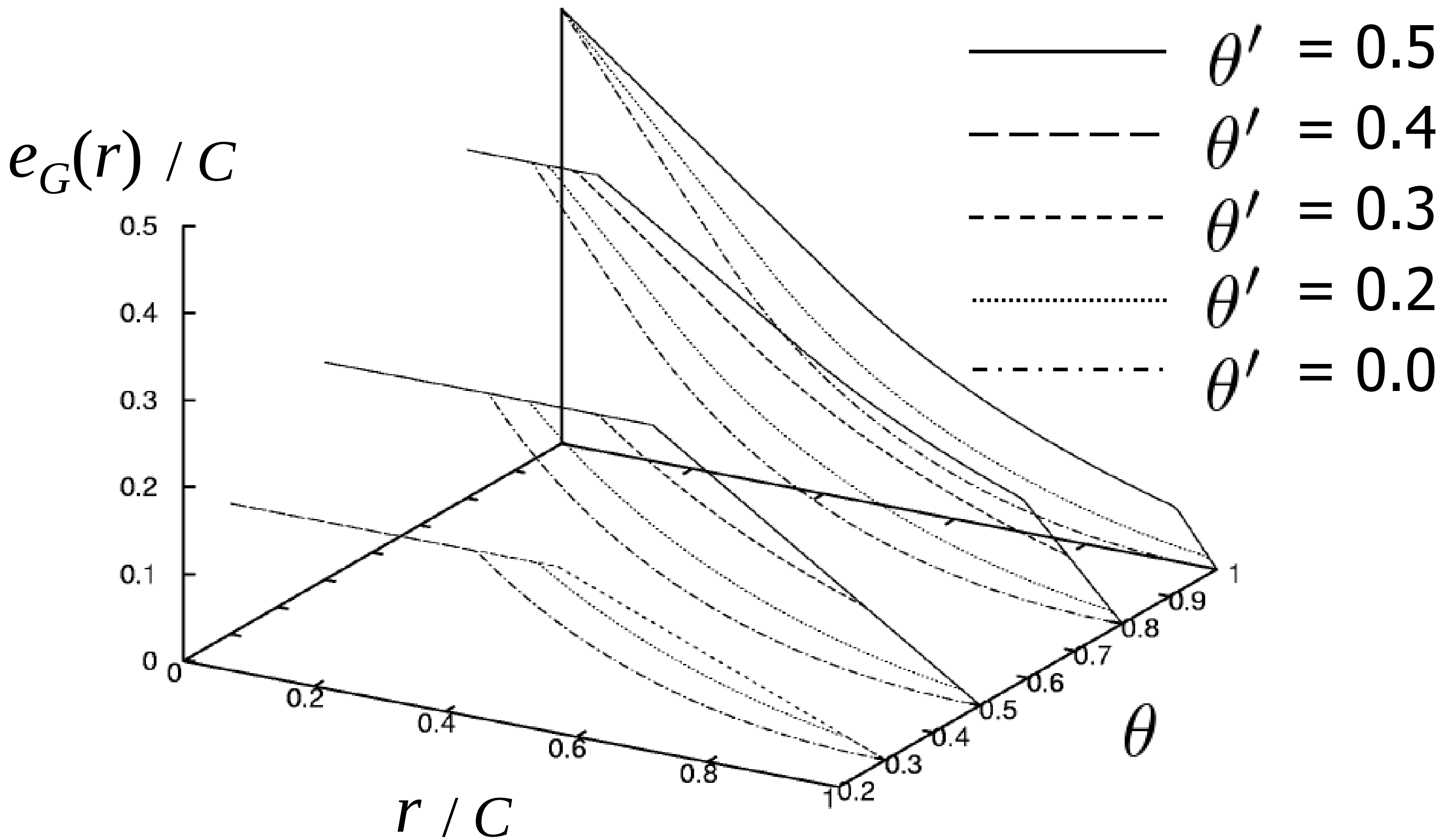
[Example 1] GTB trellis codes for Very Noisy Channel

$$e_G(r) = \min\{\theta e(r), E[(1-\theta')r], E(\theta' r)\}$$



(b) FTB Trellis codes, (1) $\theta = 0.5, \theta' = 0.5$ (2) $\theta = 0.3, \theta' = 0.3$

$$E[(1-\theta)r] \geq \theta e(r) \quad (0 < r \leq C)$$ [3]



(a) PTB trellis codes (θ' = 0.5)

(b) FTB trellis codes $(\theta = \theta')$

[Corollary 1] FTB trellis code [4]:

$$\Pr(\mathcal{E}) \leq \exp[-N\theta e(r)] \quad (0 < r < C) \quad (0 \leq \theta \leq \tfrac{1}{2}) \tag{17}$$

$$G \sim q^{2v} = \exp[2N\theta r] \tag{18}$$

[Corollary 2] upper bounds on $\Pr(\mathcal{E})$

Ordinary block codes
Terminated trellis codes
} > DT trellis codes

	error exponent	decoding complexity	upper bound on Pr(･)
Terminated trellis codes [3]:	$E(R)$	q^v	$G^{-\frac{1-\theta}{\theta}\frac{E(R)}{R}}$
DT trellis codes:	$E(r)$	q^v	$G^{-\frac{1}{\theta}\frac{E(r)}{r}}$

3.2 Decoding complexity for GTB trellis codes

[Theorem 2]

$$G \sim q^{v+v'} = \exp[N(\theta + \theta')r] \quad (0 \le \theta' \le \theta \le 1) \qquad (19)$$

3.3 Upper bounds on probability error for same decoding complexity

(N, K) block code

$$G \sim \exp[NR] \tag{3}$$

$$\Pr(\mathcal{E}) \leq \exp[-E(R)]$$
$$= G^{-\frac{E(R)}{R}}$$

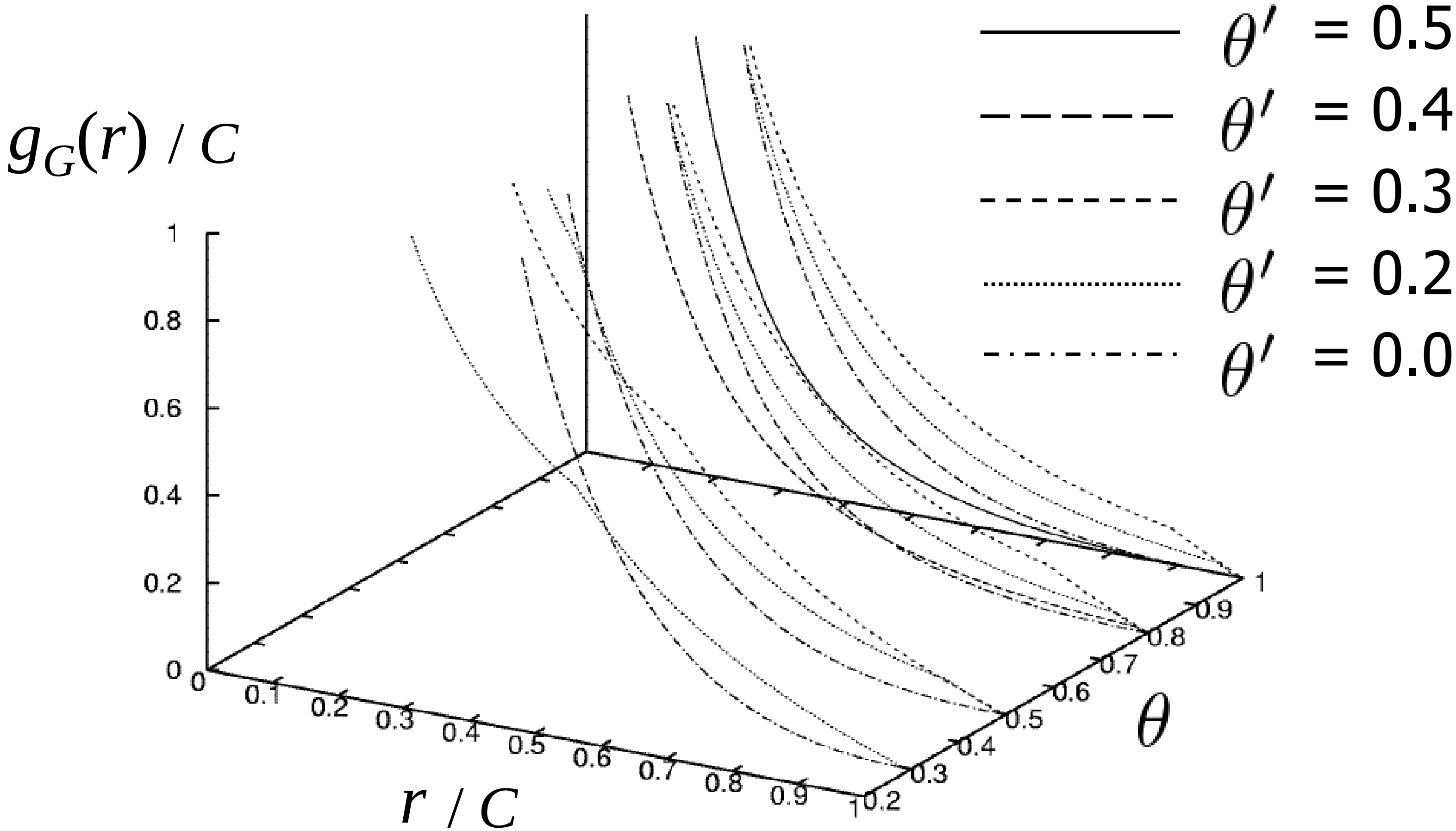
[Corollary 3] GTB trellis codes:

$$\Pr(\mathcal{E}) \leq G^{-g_G(r)} \tag{22}$$

where

$$g_G(r) = \frac{1}{(\theta + \theta')r} \min\{\theta e(r), E[(1 - \theta')r], E(\theta' r)\}$$

[Example 2] GTB trellis codes for Very Noisy Channel



(a) $1/2 < \theta \leq 1$: PTB trellis codes (θ' = 0.5 except for low rates)

(b) $0 < \theta \leq 1/2$: FTB trellis codes $(\theta = \theta')$

3. GTB trellis codes

Table 1: Asymptotic results on error exponents and decoding complexity for block codes

Block code	error exponent	decoding complexity	upper bound on Pr(·)
Ordinary block code	$E(R)$	$\exp[NR]$	$G^{-\frac{E(R)}{R}}$
Terminated trellis code	$E(R)$ [3]	q^v	$G^{-\frac{1-\theta}{\theta}\frac{E(R)}{R}}$
GTB trellis code (Theorems 1 and 2)			
DT trellis code ($\theta' = 0$)	$E(r)$ [3]	q^v	$G^{-\frac{1}{\theta}\frac{E(r)}{r}}$
PTB trellis code ($0 < \theta' < \theta$)	$e_G(r)$	$q^{v+v'}$	$G^{-\frac{1}{\theta+\theta'}\frac{e_G(r)}{r}}$ †
FTB trellis code ($\theta' = \theta$)	$e_G(r)$	q^{2v}	$G^{-\frac{1}{2\theta}\frac{e_G(r)}{r}}$ †

4. Generalized version of concatenated codes with GTB trellis inner codes

[4] **Example 2** The case of $\theta = \frac{1}{2}$ gives the largest error exponent for the code $\mathcal{C}_T$ with the same overall decoding complexity for the code $\mathcal{C}_T$ and for the code $\mathcal{C}$. On a very noisy channel, the error exponent for the code $\mathcal{C}_T$ is larger than that for the code $\mathcal{C}$, except for $0 \leq R_0 \leq 0.06\,C$. Substitution of (D.1) and (D.2) into (25) and (21), respectively, gives Fig. 2. □

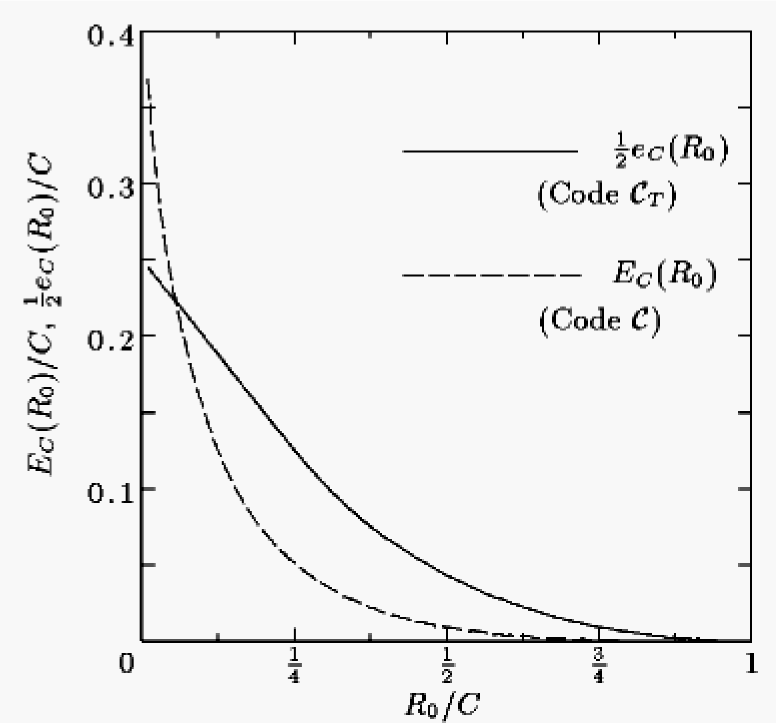


Fig.2: Error exponents for code $\mathcal{C}$ and code $\mathcal{C}_T$ for very noisy channel

Generalized version of concatenated codes with GTB trellis codes

[Lemma 1] [4, 5]: $(J = 1)$

$$\Pr(\mathcal{E}) \leq \exp[-N_0 \theta e_C(R_0)] \quad \left(0 < \theta \leq \frac{1}{2}, 0 \leq R_0 < C\right)$$

where

$$e_C(R_0) = \max_{0<r<C} \left(1 - \frac{R_0}{r}\right) e(r)$$

$$G_0 = O(N_0^2 \log^2 N_0) \quad \left(0 < \theta \leq \frac{1}{2}\right)$$

	error exponent	decoding complexity	upper bound on Pr(·)
Terminated trellis codes [6]:	$E(R)$	q^v	$G^{-\frac{1-\theta}{\theta}\frac{E(R)}{R}}$
DT trellis codes:	$E(r)$	q^v	$G^{-\frac{1}{\theta}\frac{E(r)}{r}}$

Length of outer code $n = q^{\frac{u}{J}}$,

Decoding complexity of inner code $O(n^{1+J(\theta+\theta')} \log^2 n)$

[Theorem 3] Generalized version of concatenated codes $C^{(J)}$ $(J > 1)$[6]:

With the DT trellis inner codes

$$\Pr(\mathcal{E}) \le \exp[-N_0 E_C^{(J)}(R_0)] \quad (0 \le R_0 \le C)$$

where

$$E_C^{(J)}(R_0) = \max_R \left(1 - \frac{R_0}{R}\right) \frac{J}{\sum_{j=1}^{J} \left[\frac{E(R)}{E(\frac{jR}{J})}\right]} E(R)$$

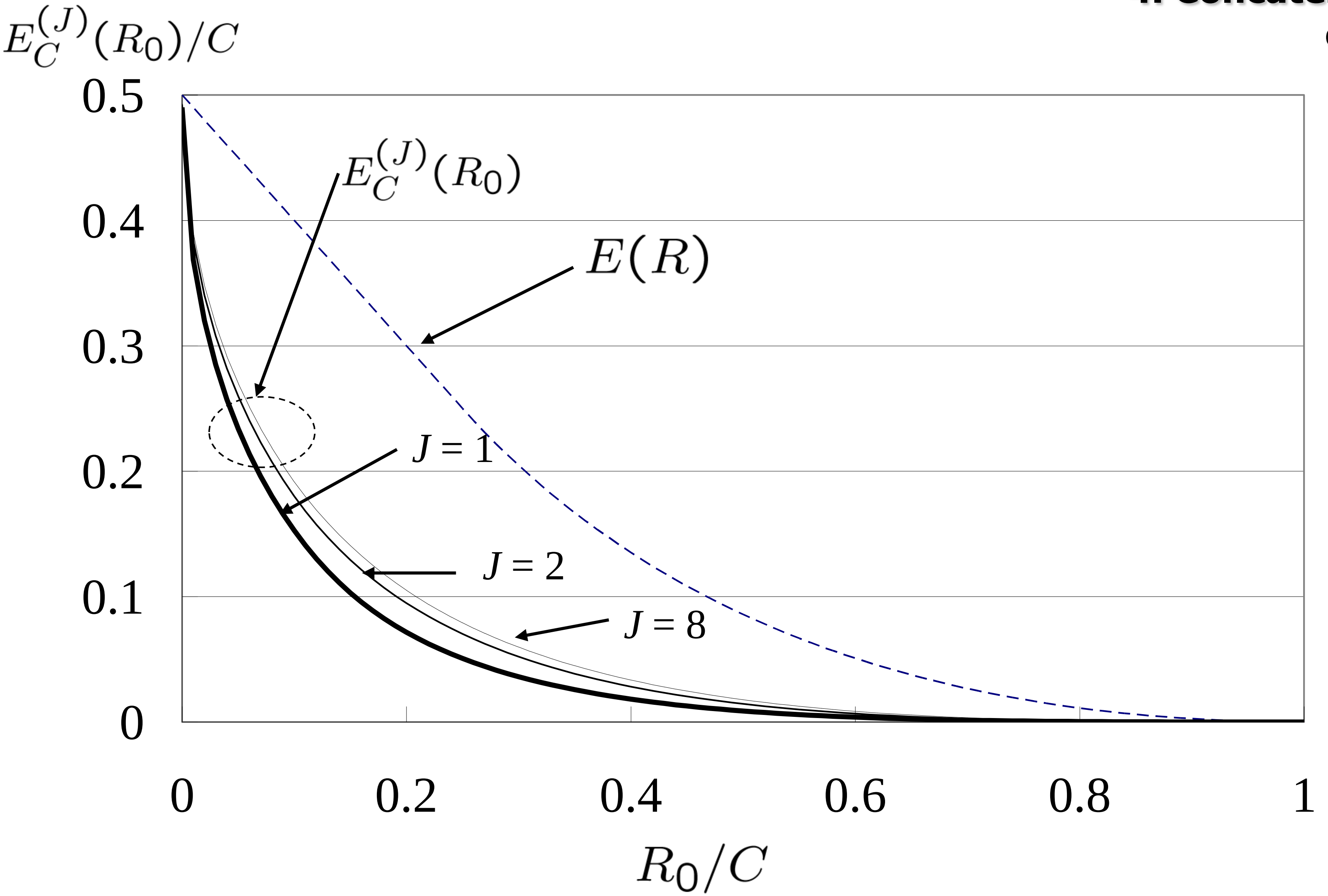


Figure 6: (a) Code $C^{(J)}$ with DT trellis inner codes over a very noisy channel

[Theorem 4] Code $C^{(J)}$

With the GTB trellis inner codes

$$\Pr(\mathcal{E}) \leq \exp[-N_0 e_C^{(J)}(R_0)] \tag{29}$$

where

$$e_C^{(J)}(R_0) = \max_r (1 - \frac{R_0}{r}) \frac{J}{\sum_{j=1}^{J} [\frac{e_G(r)}{e_G(\frac{jr}{J})}]} e_G(r). \tag{30}$$

Decoding complexity

$$G(N_0) = \begin{cases} O(N_0^2 \log^2 N_0), & J = 1; \\ O(N_0^{1+J(\theta+\theta')} \log^{1-J(\theta+\theta')} N_0), & J \geq 2, \end{cases} \tag{31}$$

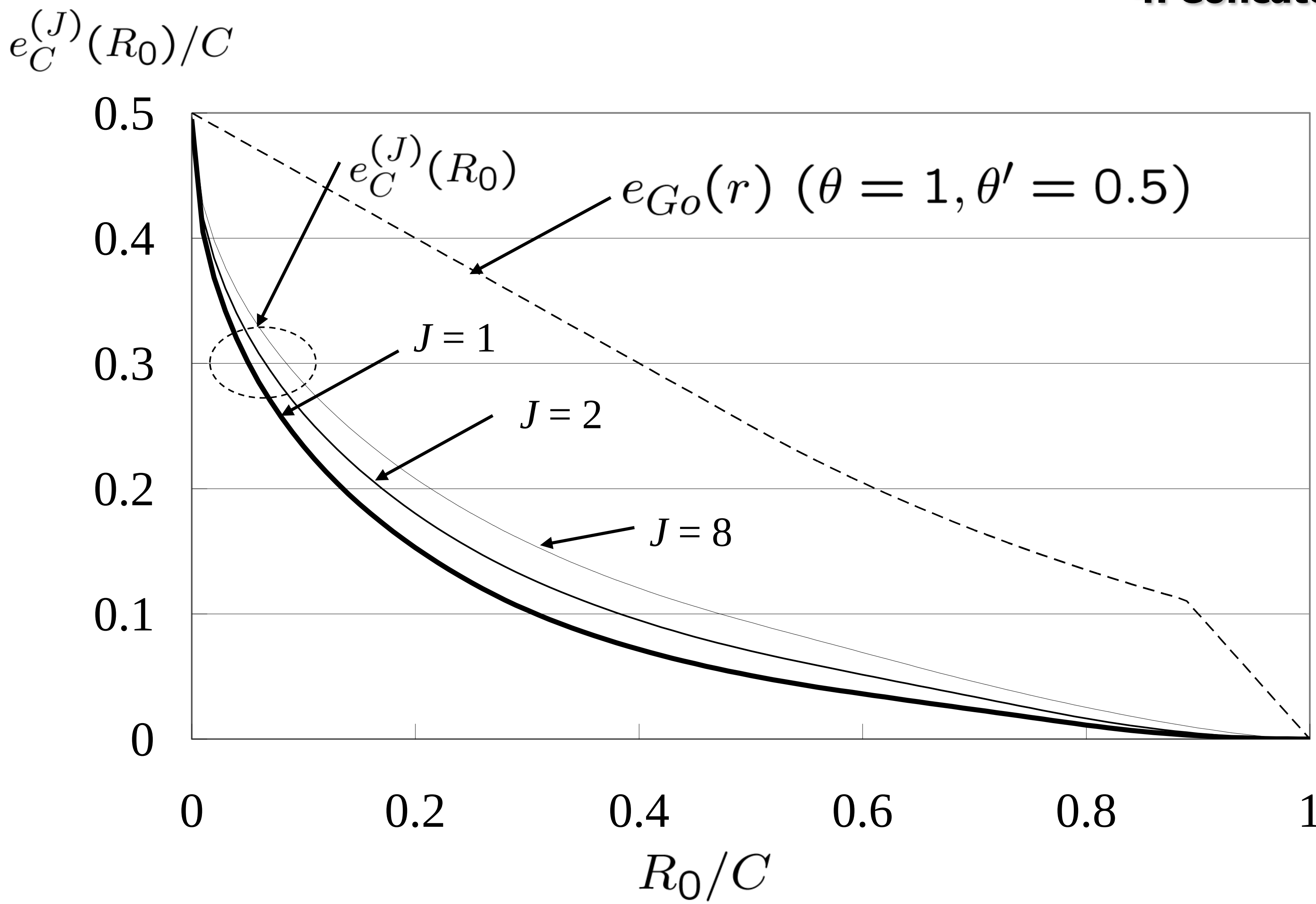


Figure 6: (b) Code $C^{(J)}$ with PTB trellis inner codes over a very noisy channel

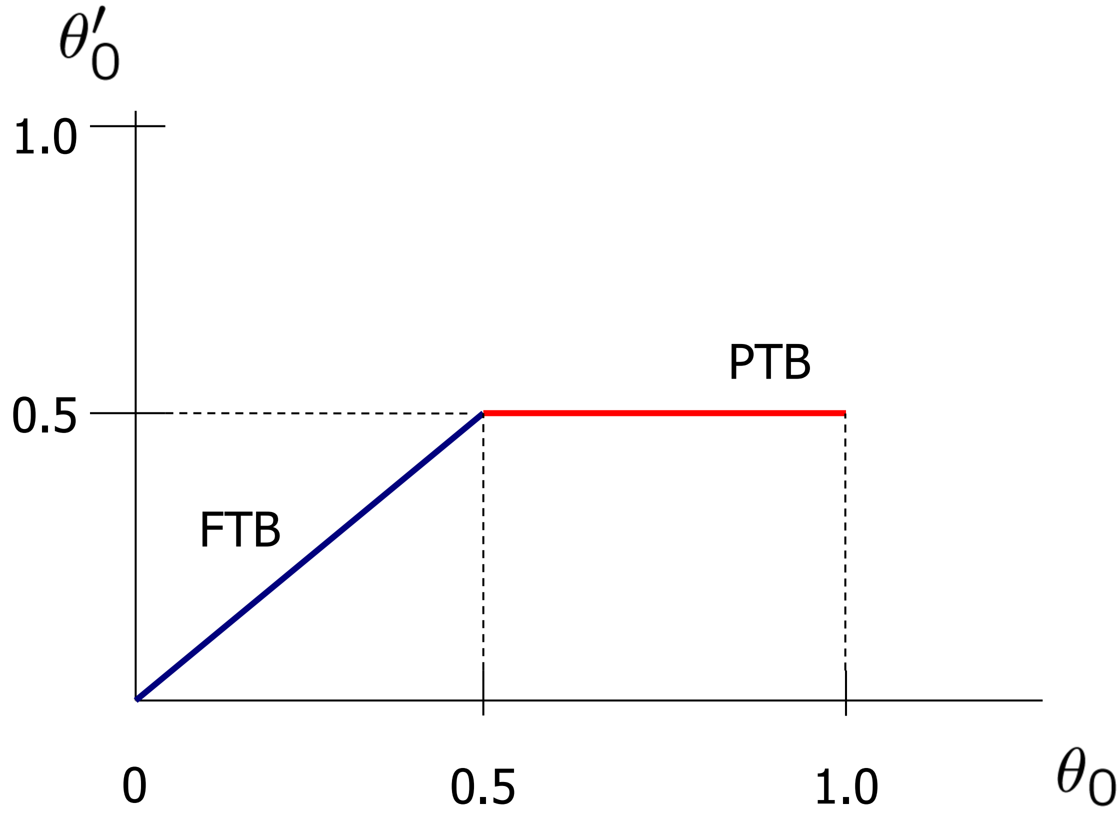


Figure 7: Optimum parameters θ_0 and θ'_0 over a very noisy channel

5. Concluding remarks

(1) GTB trellis codes over Very Noisy Channel

$1/2 < \theta \leq 1 \ \ (\theta' = 0.5):$ PTB

$0 < \theta \leq 1/2 \ \ (\theta = \theta'):$ FTB

(2) Generalized version of concatenated codes $C^{(J)}$

- Exponent + Decoding complexity
- Terminated trellis codes = DT trellis codes

Inner codes: $O(nN^2 e^{NR}) = O(n^2 \log^2 n)$

Outer codes: GMD $O(nd \log^4 n) = O(n^2 \log^4 n) \ \ (J=1)$

E-O $O(n \log^4 n) \ \ (J=1)$

Over-all decoding complexity is dominated by that of inner codes!

5. Concluding remarks

(3) Generalized version of concatenated codes $C^{(J)}$

GTB trellis inner codes --- PTB trellis codes

・ Linear error exponent + large decoding complexity

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